

# TOPOLOGICAL NATURE OF FU-KANE-MELE INVARIANTS

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**ABSTRACT.** Condensed matter electronic systems endowed with an odd time-reversal symmetry (TRS) (a.k.a. class AII topological insulators) show topologically protected phases which are described by an invariant known as Fu-Kane-Mele index. The construction of this invariant, in its original form, is specific for electrons in a periodic background and is not immediately generalizable to other interesting physical models where different forms of TRS also play a role. By exploiting the fact that system with an odd TRS (in absence of disorder) can be classified by “Quaternionic” vector bundles, we introduce a “Quaternionic” topological invariant, called FKMM-invariant, which generalizes and explains the topological nature of the Fu-Kane-Mele index. We show that the FKMM-invariant is a universal characteristic class which can be defined for “Quaternionic” vector bundles in full generality, independently of the particular nature of the base space. Moreover, it suffices to discriminate among different topological phases of system with an odd TRS in low dimension. As a particular application we describe the complete classification over a big class of low dimensional involutive spheres and tori. We also compare our classification with recent results concerning the description of topological phases for two-dimensional adiabatically perturbed systems.

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## 1. INTRODUCTION

In its simplest incarnation, a *Topological Quantum System* (TQS) is a continuous matrix-valued map

$$X \ni x \mapsto H(x) \in \text{Mat}(\mathbb{C}^N) \quad (1.1)$$

defined on a topological space  $X$  sometimes called *Brillouin zone*. Although a precise definition of TQS requires some more ingredients (see [DG1, DG2, DG3, DG4]), one can certainly state that the most relevant feature of these systems is the nature of the spectrum which is made by continuous (energy) bands. It is exactly this peculiar band structure which may encode information that are of topological nature. The study and the classification of the topological properties of TQS have recently risen to the level of “hot topic” in mathematical physics because its connection with the study of *topological insulators* in condensed matter (we refer to the two reviews [HK] and [AF] for a modern overview about topological insulators and an updated bibliography of the most relevant publications on the subject). However, systems like (1.1) are ubiquitous in the mathematical physics and are not necessarily linked with problem coming from condensed matter.

For the sake of simplicity we consider here a “concrete” realizations of a TQS of type

$$H(x) := \sum_{j=0}^{2N} F_j(x) \Sigma_j . \quad (1.2)$$

where  $\{\Sigma_0, \dots, \Sigma_{2N}\} \in \text{Mat}(\mathbb{C}^{2^N})$  are a (non-degenerate) irreducible representation of the *complex Clifford algebra*  $\text{Cl}_{\mathbb{C}}(2N)$  and the real-valued functions  $F_j : X \rightarrow \mathbb{R}$ , with  $j = 0, 1, \dots, 2N$ , are assumed to be continuous. Under the *gap condition*

$$Q(x) := \sum_{j=0}^{2N} F_j(x)^2 > 0 \quad (1.3)$$

one can associate the negative (or positive) part of the band spectrum of  $x \mapsto H(x)$  with complex vector bundle  $\mathcal{E} \rightarrow X$  of rank  $2^{N-1}$ , sometimes called *Bloch bundle*. For the details of this identification we refer to Section 5 as well as to [DG1, Section 2] or [DL, Section 4]. The considerable consequence of the duality between gapped TQS’s and spectral Bloch bundles is that one can classify the possible topological phases of the TQS by means of the elements of the set  $\text{Vec}_{\mathbb{C}}^{2^{N-1}}(X)$  given by the isomorphism classes of all rank  $2^{N-1}$  vector bundles over  $X$ . Therefore, in this general setting, the classification problem for the topological phases of a TQS like (1.2) can be traced back to a classic problem in topology which has been elegantly solved in [Pe]. For instance, in the (physically relevant) case of a *low dimensional* base space  $X$ , the complete classification of the topological phases of (1.2) is provided by

$$\text{Vec}_{\mathbb{C}}^{2^{N-1}}(X) \stackrel{c_1}{\cong} H^2(X, \mathbb{Z}) , \quad \dim(X) \leq 3 \quad (1.4)$$

where on the right-hand side one has the second singular cohomology group of  $X$  and the map  $c_1$  is the *first Chern class*.

The problem of the classification of the topological phases becomes more interesting, and challenging, when the TQS is constrained by the presence of certain symmetries or *pseudo-symmetries*. Among the latter, the *time-reversal symmetry* (TRS) attracted recently a considerable interest in mathematical physics community. A system like (1.2) is said to be time-reversal symmetric if there is an *involution*  $\tau : X \rightarrow X$  on the base space and an *anti-unitary* map  $\Theta$  such that

$$\begin{cases} \Theta H(x) \Theta^* = H(\tau(x)) , & \forall x \in X \\ \Theta^2 = \epsilon \mathbb{1}_{2^N} & \epsilon = \pm 1 . \end{cases} \quad (1.5)$$

The case  $\epsilon = +1$  corresponds to an even (or bosonic) TRS. In this event, the spectral vector bundle  $\mathcal{E}$  turns out to be equipped with an additional structure named “*Real*” by M. F. Atiyah in [At1]. Therefore, in the presence of an even TRS the classification problem of the topological phases is

reduced to the study of the set  $\text{Vec}_{\mathfrak{R}}^{2^{N-1}}(X, \tau)$  of isomorphism classes of rank  $2^{N-1}$  vector bundles over  $X$  endowed with a “Real” structure. This problem has been analyzed and solved in [DG1]. In particular, in the low dimensional case one has that

$$\text{Vec}_{\mathfrak{R}}^{2^{N-1}}(X, \tau) \stackrel{c_1^{\mathfrak{R}}}{\simeq} H_{\mathbb{Z}_2}^2(X, \mathbb{Z}(1)) , \quad \dim(X) \leq 3 . \quad (1.6)$$

Here, in the right-hand side there is the equivariant Borel cohomology with local coefficient system  $\mathbb{Z}(1)$  of the involutive space  $(X, \tau)$  (see Section 2.3 and references therein for more details) and the isomorphism is given by *first “Real” Chern class*  $c_1^{\mathfrak{R}}$  introduced for the first time by B. Kahn in [Kah] (see also [DG1, Section 5.6]). Note the close similarity between the equations (1.4) and (1.6).

The case  $\epsilon = -1$  describes an odd (or fermionic) TRS. Also in this situation the spectral vector bundle  $\mathcal{E}$  acquires an additional structure called “*Quaternionic*” (or symplectic [Du]) and the topological phases of a TQS with an odd TRS turn out to be labelled by the set  $\text{Vec}_{\mathfrak{Q}}^{2^{N-1}}(X, \tau)$  of isomorphism classes of rank  $2^{N-1}$  vector bundles over  $X$  with “Quaternionic” structure.

The study of systems with an odd TRS is more interesting, and for several reasons also harder, than the case of an even TRS. Historically, the fame of these fermionic systems begins with the seminal papers [KM, FKM] by L. Fu, C. L. Kane and E. J. Mele. The central result of these works is the interpretation of a physical phenomenon called *Quantum Spin Hall Effect* as the evidence of a non-trivial topology for TQS constrained by an odd TRS. Specifically, the papers [KM, FKM] are concerned about the study of systems like (1.2) (with  $N = 2$ ) where the base space is a torus of dimension 2 or 3 endowed with a “time-reversal” involution. These involutive *TR-tori* are the quotient spaces  $(\mathbb{R}/\mathbb{Z})^d$  endowed with the quotient involution defined on  $\mathbb{R}$  by  $x \rightarrow -x$  (we denote these spaces with  $\mathbb{T}^{0,d,0}$  according to a general notation which will be clarified in equation (1.10)). The distinctive aspect of the spaces  $\mathbb{T}^{0,d,0}$  is the existence of a *fixed point set* formed by  $2^d$  isolated points. The latter plays a crucial role in the classification scheme proposed in [KM, FKM] where the different topological phases are distinguished by the signs that a particular function  $\mathfrak{d}_{\mathcal{E}}$  defined by the Bloch-bundle (essentially the inverse of a normalized pfaffian) takes on the  $2^d$  points fixed by the involution. These numbers are usually known as *Fu-Kane-Mele indices*.

In the last years the problem of the topological classification of systems with an odd TRS has been discussed with several different approaches. From one hand, there are classification schemes based on *K-theory* [Ki, FM] or *equivariant homotopy* techniques [KG, KZ] which are extremely general. In the opposite side there are constructive procedures based on the interpretation of the topological phases as *obstructions* for the construction of trivial time-reversal symmetric frames (see [GP] for the case  $\mathbb{T}^{0,2,0}$  and [FMP] for the generalization to the case  $\mathbb{T}^{0,3,0}$ ). All these approaches, in our opinion, present some limitations. The *K-theory* is unable to distinguish the “spurious phases” (possibly) present outside of the *stable rank* regime. The homotopy calculations are non-algorithmic and usually extremely hard. The recipes for the “handmade” construction of trivial frames are strongly dependent of the specific form of the involutive base space and thus are difficult to generalize to other spaces and higher dimensions. Finally, none of these approaches clearly identifies the invariant which labels the different phases as a *topological class*, as it happens in the cases of systems with broken TRS (*cf.* eq. (1.4)) or with even TRS (*cf.* eq. (1.6)).

To overcome these deficiencies we developed in [DG2] an alternative classification scheme based on a cohomological description. By adapting an idea of M. Furuta, Y. Kametani, H. Matsue, and N. Minami [FKMM], we introduced a cohomological class  $\kappa$  called *FKMM-invariant*, that we have shown to be a characteristic class for the category of “Quaternionic” vector bundles. In the low dimensional case this invariant provides an *injection*

$$\text{Vec}_{\mathfrak{Q}}^{2^{N-1}}(X, \tau) \xrightarrow{\kappa} H_{\mathbb{Z}_2}^2(X|X^{\tau}, \mathbb{Z}(1)) , \quad \dim(X) \leq 3 . \quad (1.7)$$

The meaning of the cohomology group which appears in right-hand side can be guessed by looking at the exact sequence (cf. Section 2.3)

$$\dots \longrightarrow [X^\tau, \{\pm 1\}] \longrightarrow H_{\mathbb{Z}_2}^2(X|X^\tau, \mathbb{Z}(1)) \longrightarrow \text{Pic}_{\mathfrak{R}}(X, \tau) \xrightarrow{r} \text{Pic}_{\mathbb{C}}(X^\tau) \longrightarrow \dots$$

where  $[X^\tau, \{\pm 1\}]$  is the set of homotopy classes of maps from  $X^\tau$  into  $\{\pm 1\}$ ,  $\text{Pic}_{\mathfrak{R}}(X, \tau) = \text{Vec}_{\mathfrak{R}}^1(X, \tau)$  is the Picard group of the “Real” line bundles over  $(X, \tau)$ ,  $\text{Pic}_{\mathbb{C}}(X^\tau) \simeq \text{Pic}_{\mathfrak{R}}(X^\tau, \tau)$  is the Picard group of the complex line bundles over  $X^\tau$  and  $r$  is the restriction map.

Despite its similarity with the equations (1.4) and (1.6), the equation (1.7) cannot be considered completely satisfactory, at least as derived in [DG2, Theorem 1.1]. The most dramatic weakness of the (1.7) in its original form is that it has been proved only under the hypothesis of  $(X, \tau)$  being an FKMM-space [DG2, Definition 1.1]. This is a strong restriction since: (i) it excludes involutive spaces with fixed point sets  $X^\tau$  of dimension *bigger than zero* and involutive spaces with a *free involution* (i. e.  $X^\tau = \emptyset$ ); (ii) it introduces by “brute force” the condition  $\text{Pic}_{\mathfrak{R}}(X, \tau) = 0$  which implies the isomorphism [DG2, Lemma 3.1]

$$H_{\mathbb{Z}_2}^2(X|X^\tau, \mathbb{Z}(1)) \simeq [X^\tau, \{\pm 1\}]/[X, \mathbb{U}(1)]_{\mathbb{Z}_2} \quad (1.8)$$

where  $[X, \mathbb{U}(1)]_{\mathbb{Z}_2}$  denotes the set of the homotopy classes of  $\mathbb{Z}_2$ -equivariant maps from  $(X, \tau)$  into  $\mathbb{U}(1)$  endowed with the involution given by the complex conjugation. Clearly, the isomorphism (1.8) fails in general for involutive spaces  $(X, \tau)$  which are not of FKMM-type. On the other hand, (1.8) is also the principal ingredient to link the FKMM-invariant  $\kappa$  with the (strong) Fu-Kane-Mele index [DG2, Theorem 4.2] through the well-known formula

$$\kappa(\mathcal{E}) = \prod_{x \in X^\tau} \mathfrak{d}_{\mathcal{E}}(x), \quad (X, \tau) \text{ an FKMM-space.} \quad (1.9)$$

There is also a second, more delicate, reason which makes (1.7) weaker than (1.4) or (1.6). In fact (1.7) provides, in general, only an injection while the other two are isomorphisms. This topic will be discussed in Section 4.3.

The main result of this paper is the extension of the range of validity of (1.7) to involutive spaces  $(X, \tau)$  in full generality. More precisely, we introduce a *generalized* version of the FKMM-invariant which extends the “old” invariant constructed in [DG2] to involutive spaces which are not necessary of FKMM-type. This generalization has a pay-off: The (1.7) becomes meaningful (and valid) for spaces with fixed point set of any co-dimension. This is, in our opinion, a big step forward in the theory of the classification of topological phases for systems with odd TRS. In fact, motivated by the (1.9), we can interpret  $\kappa$  as the extension of the Fu-Kane-Mele indices when  $X^\tau$  is not simply a collection of isolated points.

Before discussing some interesting consequences of the (1.7) in its generalized meaning, let us spend few words about the importance of considering spaces different from the TR-tori  $\mathbb{T}^{0,d,0}$ . If from one hand the spaces  $\mathbb{T}^{0,d,0}$  emerge naturally as quasi-momentum spaces (a.k.a. Brillouin zone) for  $d$ -dimensional periodic electronic systems, on the other hand topological quantum systems of type (1.2) are ubiquitous in mathematical physics and are not necessarily related to model in condensed matter (see e. g. the rich monograph [BMKNZ]). Usually, the space  $X$  plays the role of a *configuration* space for parameters which describe an *adiabatic* action of external fields on a system governed by the instantaneous Hamiltonian  $H(x)$ . The phenomenology of these systems is enriched by the presence of certain symmetries like a TRS as in (1.5). Models of *adiabatic* topological systems of this type have been recently investigated in [CDF, GR]. In particular, in the second of these works [GR] the authors consider the adiabatically perturbed dynamics of a *classic rigid rotor* and a *classical particle on a ring*. In the first case the classical phase space turns out to be a two-dimensional sphere endowed with an antipodal involution induced by the TRS (we use  $\mathbb{S}^{0,3}$  for this space). In the second case the phase space is a two dimensional torus  $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$  where the TRS induces an time-reversal involution only on one of its factors (we denote this space by  $\mathbb{T}^{1,1,0}$ ). The main result of [GR] consists in the classification of “Quaternionic” vector bundles over the involutive spaces  $\mathbb{S}^{0,3}$  and  $\mathbb{T}^{1,1,0}$  and

it is based on the analysis of the obstruction for the “handmade” construction of a trivial frame. As already commented above, this technique is hard (and tricky) to extend to higher dimensions and other involutive spaces. Conversely, the classification provided by the map (1.7) turns out to be extremely effective and versatile. In fact it is algorithmic! As a matter of fact the formula (1.7) allows us to classify “Quaternionic” vector bundles over a big class of involutive spheres and tori up to dimension three extending, in this way, the results in [GR]. The cost for such a classification amounts to just a little more than the (algorithmic) calculation of the cohomology group in (1.7). We postpone a synthesis of the results of this classification at the end of this introductory section.

The map (1.7) can be used also to obtain information about the possibility of non trivial topological states for systems of type (1.2) independently of the details of the functions  $F_j$ . For instance, the study of the case  $N = 2$  in Section 5 leads to the following general result:

**Theorem 1.1** (Necessary condition for non-trivial phases). *Consider a topological quantum system of type 1.2 with  $N = 2$  and endowed with an odd TRS of type (1.5). A necessary condition for the existence of non-trivial phases is that  $F_2$  and at least two of the functions  $\{F_0, F_1, F_3, F_4\}$  must be not identically zero.*

For sake of precision, one has to refer the statement of the theorem above to the definitions and notations introduced in Section 5. Note that this result is just a (partial) summary of the content of Proposition 5.2 and Proposition 5.3.

This paper is organized as follows: In **Section 2** we construct the *generalized* version of the FKMM-invariant and we prove that this is a characteristic class for the category of “Quaternionic” vector bundles. Moreover, we show that it reduces to the “old version” of the FKMM-invariant described in [DG2], and so to the Fu-Kane-Mele indices in the case of spaces with a discrete number of fixed points. In **Section 4** we use the “new” FKMM-invariant to classify “Quaternionic” vector bundles over low dimensional involutive spaces. We provide a complete classification for “Quaternionic” line bundles which exists in case of free-involution spaces. We also discuss the classification over involutive spheres and tori. **Section 5** is devoted to the study of TQS of type (1.2) with  $N = 2$ . Finally, **Appendices A, B** and **C** contain all the cohomology computations needed in the paper. These calculations are technical, but also of general validity and may also be useful in other areas of the mathematical research.

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**Resumé of the classification over involutive spheres and tori.** Let us fix some notations: The *involutive sphere* of type  $\mathbb{S}^{p,q} := (\mathbb{S}^{p+q-1}, \theta_{p,q})$  is defined as the sphere of dimension  $d := p + q - 1$

$$\mathbb{S}^d := \{(k_0, k_1, \dots, k_d) \in \mathbb{R}^{d+1} \mid k_0^2 + k_1^2 + \dots + k_d^2 = 1\}$$

endowed with the involution

$$\theta_{p,q}(k_0, k_1, \dots, k_{p-1}, k_p, k_{p+1}, \dots, k_{p+q-1}) := (k_0, k_1, \dots, k_{p-1}, -k_p, -k_{p+1}, \dots, -k_{p+q-1})$$

The space  $\mathbb{S}^{d+1,0}$  coincide with the sphere  $\mathbb{S}^d$  endowed with the *trivial involution* fixing all points. On the opposite side one has the space  $\mathbb{S}^{0,d+1}$  which is the sphere  $\mathbb{S}^d$  endowed with the *antipodal* (or *free*) *involution*. The space  $\mathbb{S}^{1,d}$  (denoted with the symbol  $\tilde{\mathbb{S}}^d$  in [DG1, DG2, DG3]) coincide with the “Brillouin zone” of condensed matter systems invariant under continuous translations and subjected to a TRS. For this reason one often refers to  $\theta_{1,d}$  as a *TR-involution*. For more details, the reader can refer to [DG1, Section 2]. The classification of “Quaternionic” vector bundles over these spaces in low dimension is summarized in Table 1.1. We notice that the case  $\mathbb{S}^{0,3}$  studied in [GR] is also included.

With multiple products of involutive one-dimensional spheres one can define several types of *involutive tori*. By using as building blocks the three involutive space  $\mathbb{S}^{2,0}$ ,  $\mathbb{S}^{1,1}$  and  $\mathbb{S}^{0,2}$  one can define the



involutive torus of type  $(a, b, c)$  by

$$\mathbb{T}^{a,b,c} := \underbrace{\mathbb{S}^{2,0} \times \dots \times \mathbb{S}^{2,0}}_{a\text{-times}} \times \underbrace{\mathbb{S}^{1,1} \times \dots \times \mathbb{S}^{1,1}}_{b\text{-times}} \times \underbrace{\mathbb{S}^{0,2} \times \dots \times \mathbb{S}^{0,2}}_{c\text{-times}} \quad a, b, c \in \mathbb{N} \cup \{0\}. \quad (1.10)$$

Topologically this is a torus of dimension  $d = a + b + c$  endowed with the natural product involution. The space  $\mathbb{T}^{d,0,0}$  coincide with the torus  $\mathbb{T}^d$  endowed with the *trivial involution* fixing all points. The space  $\mathbb{T}^{0,d,0}$  (denoted with the symbol  $\tilde{\mathbb{T}}^d$  in [DG1, DG2, DG3]) coincide with the “Brillouin zone” of condensed matter systems invariant under  $\mathbb{Z}^d$ -translations and subjected to a TRS (for more details, we refer to [DG1, Section 2]). Notice that  $c \neq 0$  implies that the involution acts freely. For a fixed dimension  $d$  there are  $\frac{1}{2}(d+1)(d+2)$  possible combinations for  $a+b+c = d$  but only  $2d+1$  inequivalent involutive tori due to Proposition 4.9 which states the equivalence

$$\mathbb{T}^{a,b,c} \simeq \mathbb{T}^{a+c-1,b,1} \quad c \geq 2. \quad (1.11)$$

The classification in the case of tori with fixed point ( $c = 0$ ) up to dimension three is described in Table 1.2. Note that also the case  $\mathbb{T}^{1,1,0}$  discussed in [GR] is included.

$p + q \leq 4$	$q = 0$	$q = 1$	$q = 2$	$q = 3$	$q = 4$
$\text{Vec}_{\mathbb{Q}}^{2m+1}(\mathbb{S}^{0,q})$	$\emptyset$	0	0	$2\mathbb{Z} + 1$	$\emptyset$
$\text{Vec}_{\mathbb{Q}}^{2m}(\mathbb{S}^{0,q})$	$\emptyset$	0	0	$2\mathbb{Z}$	0
$\text{Vec}_{\mathbb{Q}}^{2m}(\mathbb{S}^{1,q})$	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\dots$
$\text{Vec}_{\mathbb{Q}}^{2m}(\mathbb{S}^{2,q})$	0	$2\mathbb{Z}$	0	$\dots$	
$\text{Vec}_{\mathbb{Q}}^{2m}(\mathbb{S}^{3,q})$	0	0	$\dots$		
$\text{Vec}_{\mathbb{Q}}^{2m}(\mathbb{S}^{4,q})$	0	$\dots$			

TABLE 1.1. Complete classification of “Quaternionic” vector bundles over involutive spheres of type  $\mathbb{S}^{p,q}$  in low dimension  $d = p + q - 1 \leq 3$ . The symbol  $\emptyset$  (empty set) denotes the impossibility of constructing “Quaternionic” vector bundles. The symbol 0 denotes the existence of a unique element which can be identified with the trivial product bundle in the even rank case. In the odd rank case, only possible over the free-involution spheres  $\mathbb{S}^{0,q}$ , the notion of trivial bundle is not well defined, and even when there is only one representative (e.g. the cases  $q = 1, 2$ ) this is not given by a product bundle with a product “Quaternionic” action. The classification by integers (even, odd) is attained by looking at the first Chern class of the underlying complex vector bundle. The  $\mathbb{Z}_2$  classification is given by the FKMM-invariant. The results summarized in this table are discussed in detail in Section 4.4.

The case with free involution (the equivalence (1.11) allows to consider only the case  $c = 1$ ) is summarized in the Table 1.3.

**Remark 1.2** (Flip involution). Over  $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$  there is another interesting involution which is essentially different from the involutions described by (1.10). Consider the *flip* map  $\gamma : (k, k') \mapsto (k', k)$  and the associated involutive space  $\mathbb{T}_{\text{flip}}^2 := (\mathbb{T}^2, \gamma)$ . This space has a non-empty fixed point set  $(\mathbb{T}_{\text{flip}}^2)^\gamma \simeq \mathbb{S}^1$  given by the *diagonal* subset of points  $(k, k)$  and therefore it admits only even rank “Quaternionic” vector bundles. We anticipate that

$$\text{Vec}_{\mathbb{Q}}^{2m}(\mathbb{T}_{\text{flip}}^2) \stackrel{c_1}{\simeq} 2\mathbb{Z}$$

although we will study this case in a separated paper due to its relevance with the physics of a system of two identical one-dimensional particles. ◀

$a + b \leq 3, c = 0$	$a = 0$	$a = 1$	$a = 2$	$a = 3$
$\text{Vec}_{\mathbb{Q}}^{2m}(\mathbb{T}^{a,0,0})$	$\emptyset$	0	0	0
$\text{Vec}_{\mathbb{Q}}^{2m}(\mathbb{T}^{a,1,0})$	0	$2\mathbb{Z}$	$(2\mathbb{Z})^2$	$\dots$
$\text{Vec}_{\mathbb{Q}}^{2m}(\mathbb{T}^{a,2,0})$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \oplus (2\mathbb{Z})^2$	$\dots$	
$\text{Vec}_{\mathbb{Q}}^{2m}(\mathbb{T}^{a,3,0})$	$\mathbb{Z}_2^4$	$\dots$		

TABLE 1.2. Complete classification of “Quaternionic” vector bundles over involutive tori of type  $\mathbb{T}^{a,b,0}$  in low dimension  $d = a + b \leq 3$ . The existence of fixed points implies that only even rank “Quaternionic” vector bundles are admissible. The classification by (even) integers is attained by looking at the first Chern class of the underlying complex vector bundle. The  $\mathbb{Z}_2$  classification is given by the FKMM-invariant. The results summarized in this table are discussed in detail in Section 4.5.

$a + b \leq 2, c = 1$	$a = 0$	$a = 1$	$a = 2$
$\text{Vec}_{\mathbb{Q}}^m(\mathbb{T}^{a,0,1})$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$
$\text{Vec}_{\mathbb{Q}}^m(\mathbb{T}^{a,1,1})$	$2\mathbb{Z}$	$\mathbb{Z}_2 \oplus (2\mathbb{Z})^2$	$\dots$
$\text{Vec}_{\mathbb{Q}}^m(\mathbb{T}^{a,2,1})$	$(2\mathbb{Z})^2$	$\dots$	$\dots$

TABLE 1.3. Complete classification of “Quaternionic” vector bundles over involutive tori of type  $\mathbb{T}^{a,b,1}$  in low dimension  $d = a + b + 1 \leq 3$ . The absence of fixed points allows for “Quaternionic” vector bundles of every rank (here  $m \in \mathbb{Z}$ ). The classification by (even) integers is given by the first Chern class of the underlying complex vector bundle. The  $\mathbb{Z}_2$  classification is given by the FKMM-invariant. The results summarized in this table are discussed in detail in Section 4.5.

## 2. FKMM-INVARIANT: DEFINITIONS AND PROPERTIES

At a topological level, isomorphism classes of “Quaternionic” vector bundles can be classified by the *FKMM-invariant* [FKMM, DG2]. In its original form the FKMM-invariant can be introduced only for a subfamily of “Quaternionic” vector bundles provided that certain conditions are met. The aim of this section is to drop this unpleasant restriction by introducing a more general version of the FKMM-invariant.

**2.1. Basic definitions.** In this section we recall some basic facts about the category of “Quaternionic” vector bundles and we refer to [DG2] for a more systematic presentation.

An involution  $\tau$  on a topological space  $X$  is a homeomorphism of period 2, i. e.  $\tau^2 = \text{Id}_X$ . The pair  $(X, \tau)$  will be called an *involutive space*. The *fixed point* set of the  $(X, \tau)$  is by definition  $X^\tau := \{x \in X \mid \tau(x) = x\}$ . Henceforth, we will assume that:

**Assumption 2.1.**  $X$  is a compact and path-connected Hausdorff space which admits the structure of a  $\mathbb{Z}_2$ -CW-complex. The dimension  $d$  of  $X$  is the maximal dimension of its cells.

For the sake of completeness, let us recall that an involutive space  $(X, \tau)$  has the structure of a  $\mathbb{Z}_2$ -CW-complex if it admits a skeleton decomposition given by gluing cells of different dimensions which carry a  $\mathbb{Z}_2$ -action. For a precise definition of the notion of  $\mathbb{Z}_2$ -CW-complex, the reader can refer to [DG1, Section 4.5] or [Mat, AP].

We are now in position to introduce the principal object of interest in this work.

**Definition 2.2** (“Quaternionic” vector bundles [Du, DG2]). A “Quaternionic” vector bundle, or  $\mathfrak{Q}$ -bundle, over  $(X, \tau)$  is a complex vector bundle  $\pi : \mathcal{E} \rightarrow X$  endowed with a homeomorphism  $\Theta : \mathcal{E} \rightarrow \mathcal{E}$  such that:

- ( $\mathfrak{Q}_1$ ) the projection  $\pi$  is equivariant in the sense that  $\pi \circ \Theta = \tau \circ \pi$ ;
- ( $\mathfrak{Q}_2$ )  $\Theta$  is anti-linear on each fiber, i. e.  $\Theta(\lambda p) = \bar{\lambda} \Theta(p)$  for all  $\lambda \in \mathbb{C}$  and  $p \in \mathcal{E}$  where  $\bar{\lambda}$  is the complex conjugate of  $\lambda$ ;
- ( $\mathfrak{Q}_3$ )  $\Theta^2$  acts fiberwise as the multiplication by  $-1$ , namely  $\Theta^2|_{\mathcal{E}_x} = -1_{\mathcal{E}_x}$ .

It is always possible to endow  $\mathcal{E}$  with a (essentially unique) Hermitian metric with respect to which  $\Theta$  is an *anti-unitary* map between conjugate fibers [DG2, Proposition 2.5].

A vector bundle *morphism*  $f$  between two vector bundles  $\pi : \mathcal{E} \rightarrow X$  and  $\pi' : \mathcal{E}' \rightarrow X$  over the same base space is a continuous map  $f : \mathcal{E} \rightarrow \mathcal{E}'$  which is *fiber preserving* in the sense that  $\pi = \pi' \circ f$  and that restricts to a linear map on each fiber  $f|_x : \mathcal{E}_x \rightarrow \mathcal{E}'_x$ . Complex vector bundles over  $X$  together with vector bundle morphisms define a category and the symbol  $\text{Vec}_{\mathbb{C}}^m(X)$  is used to denote the set of isomorphism classes of vector bundles of rank  $m$ . Also  $\mathfrak{Q}$ -bundles define a category with respect to  $\mathfrak{Q}$ -morphisms. A  $\mathfrak{Q}$ -morphism  $f$  between two “Quaternionic” vector bundles  $(\mathcal{E}, \Theta)$  and  $(\mathcal{E}', \Theta')$  over the same involutive space  $(X, \tau)$  is a vector bundle morphism commuting with the “Quaternionic” structures, i. e.  $f \circ \Theta = \Theta' \circ f$ . The set of isomorphism classes of  $\mathfrak{Q}$ -bundles of rank  $m$  over  $(X, \tau)$  will be denoted by  $\text{Vec}_{\mathfrak{Q}}^m(X, \tau)$ .

**Remark 2.3** (“Real” vector bundles). By changing condition ( $\mathfrak{Q}_3$ ) of Definition 2.2 with

( $\mathfrak{R}$ )  $\Theta^2$  acts fiberwise as the multiplication by 1, namely  $\Theta^2|_{\mathcal{E}_x} = 1_{\mathcal{E}_x}$

one ends in the category of “Real” (or  $\mathfrak{R}$ ) vector bundles. Isomorphism classes of rank  $m$   $\mathfrak{R}$ -bundles over the involutive space  $(X, \tau)$  are denoted by  $\text{Vec}_{\mathfrak{R}}^m(X, \tau)$ . For more details we refer to [DG1]. ◀

Consider the fiber  $\mathcal{E}_x \simeq \mathbb{C}^m$  over a fixed point  $x \in X^\tau$ . In this case the restriction  $\Theta|_{\mathcal{E}_x} \equiv J$  defines an *anti-linear* map  $J : \mathcal{E}_x \rightarrow \mathcal{E}_x$  such that  $J^2 = -1_{\mathcal{E}_x}$ . Then, fibers over fixed points are endowed with a *quaternionic* structure in the sense of [DG2, Remark 2.1]. This fact has an important consequence: if  $X^\tau \neq \emptyset$  then every “Quaternionic” vector bundle over  $(X, \tau)$  has necessarily even rank [DG2, Proposition 2.1]. However, odd rank  $\mathfrak{Q}$ -bundles are possible in the case of a free involution, i. e. when  $X^\tau = \emptyset$  (cf. Section 3).

The set  $\text{Vec}_{\mathfrak{Q}}^{2m}(X, \tau)$  is non-empty since it contains at least the “Quaternionic” *product bundle*  $X \times \mathbb{C}^{2m} \rightarrow X$  endowed with the product  $\mathfrak{Q}$ -structure  $\Theta_0(x, v) = (\tau(x), Q \bar{v})$  where the matrix  $Q$  is given by

$$Q := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \otimes \mathbb{1}_m = \left( \begin{array}{cc|cc|cc} 0 & -1 & & & & \\ 1 & 0 & & & & \\ \hline & & \ddots & & & \\ & & & \ddots & & \\ \hline & & & & 0 & -1 \\ & & & & 1 & 0 \end{array} \right). \quad (2.1)$$

A  $\mathfrak{Q}$ -bundle is said to be  *$\mathfrak{Q}$ -trivial* if it is isomorphic to the product  $\mathfrak{Q}$ -bundle in the category of  $\mathfrak{Q}$ -bundles.



**2.2. The determinant construction.** Let  $\mathcal{V}$  be a complex vector space of dimension  $m$ . The *determinant* of  $\mathcal{V}$  is by definition  $\det(\mathcal{V}) := \bigwedge^m \mathcal{V}$  where the symbol  $\bigwedge^m$  denotes the top exterior power of  $\mathcal{V}$  (i.e. the skew-symmetrized  $m$ -th tensor power of  $\mathcal{V}$ ). This is a complex vector space of dimension one. If  $\mathcal{W}$  is a second vector space of the same dimension  $m$  and  $T : \mathcal{V} \rightarrow \mathcal{W}$  is a linear map then there is a naturally associated map  $\det(T) : \det(\mathcal{V}) \rightarrow \det(\mathcal{W})$  which in the special case  $\mathcal{V} = \mathcal{W}$  coincides with the multiplication by the determinant of the endomorphism  $T$ . This determinant construction is a functor from the category of vector spaces to itself and, by a standard argument [Hus, Chapter 5, Section 6], induces a functor on the category of complex vector bundles over an arbitrary space  $X$ . More precisely, for each rank  $m$  complex vector bundle  $\mathcal{E} \rightarrow X$ , the associated *determinant line bundle*  $\det(\mathcal{E}) \rightarrow X$  is the rank 1 complex vector bundle with fibers

$$\det(\mathcal{E})_x = \det(\mathcal{E}_x) \quad x \in X. \quad (2.2)$$

Each local frame of sections  $\{s_1, \dots, s_m\}$  of  $\mathcal{E}$  over the open set  $\mathcal{U} \subset X$  induces the section  $s_1 \wedge \dots \wedge s_m$  of  $\det(\mathcal{E})$  which fixes a trivialization over  $\mathcal{U}$ . Given a map  $\varphi : X \rightarrow Y$  one can prove the isomorphism  $\det(\varphi^*(\mathcal{E})) \simeq \varphi^*(\det(\mathcal{E}))$  which is a special case of the compatibility between pullback and tensor product. Finally, if  $\mathcal{E} = \mathcal{E}_1 \oplus \mathcal{E}_2$  in the sense of the Whitney sum then  $\det(\mathcal{E}) = \det(\mathcal{E}_1) \otimes \det(\mathcal{E}_2)$ .

Let  $(\mathcal{E}, \Theta)$  be a rank  $m$  “Quaternionic” vector bundle over  $(X, \tau)$ . The associated determinant line bundle  $\det(\mathcal{E})$  inherits an involutive structure given by the map  $\det(\Theta)$  which acts *anti-linearly* between the fibers  $\det(\mathcal{E})_x$  and  $\det(\mathcal{E})_{\tau(x)}$  according to  $\det(\Theta)(p_1 \wedge \dots \wedge p_m) = \Theta(p_1) \wedge \dots \wedge \Theta(p_m)$ . Clearly  $\det(\Theta)^2$  is a fiber preserving map which coincides with the multiplication by  $(-1)^m$ . Hence, in view of Remark 2.3 one has the following result.

**Lemma 2.4.** *Let  $(\mathcal{E}, \Theta)$  be a rank  $m$  “Quaternionic” vector bundle over  $(X, \tau)$ . The associated determinant line bundle  $\det(\mathcal{E})$  endowed with the involutive structure  $\det(\Theta)$  is a “Real” line bundle if  $m$  is even and a “Quaternionic” line bundle if  $m$  is odd.*

Let  $(\mathcal{E}, \Theta)$  be a rank  $m$   $\mathfrak{Q}$ -bundle over  $(X, \tau)$  endowed with an equivariant Hermitian metric  $\mathfrak{m}$ . These data fix a unique Hermitian metric  $\mathfrak{m}_{\det}$  on  $\det(\mathcal{E})$  which is equivariant with respect to the structure induced by  $\det(\Theta)$ . More explicitly, if  $(p_i, q_i) \in \mathcal{E}|_X \times \mathcal{E}|_X$ ,  $i = 1, \dots, m$  then,

$$\mathfrak{m}_{\det}(p_1 \wedge \dots \wedge p_m, q_1 \wedge \dots \wedge q_m) := \prod_{i=1}^m \mathfrak{m}(p_i, q_i).$$

The line bundle  $(\det(\mathcal{E}), \det(\Theta))$  endowed with  $\mathfrak{m}_{\det}$  is trivial if and only if there exists an isometric equivariant isomorphism (in the appropriate category) with  $X \times \mathbb{C}$ . Equivalently, if and only if there exists a global equivariant section  $s : X \rightarrow \det(\mathcal{E})$  of unit length (cf. [DG1, Theorem 4.8]). Let

$$\mathbb{S}(\det(\mathcal{E})) := \{p \in \det(\mathcal{E}) \mid \mathfrak{m}_{\det}(p, p) = 1\}$$

be the *circle bundle* underlying to  $(\det(\mathcal{E}), \det(\Theta))$ . Then the triviality of  $\det(\mathcal{E})$  can be rephrased as the existence of a global equivariant section of  $\mathbb{S}(\det(\mathcal{E})) \rightarrow X$ . The following result will play a crucial role in our construction.

**Proposition 2.5** ([DG2, Lemma 3.3]). *Let  $(\mathcal{E}, \Theta)$  be a “Quaternionic” vector bundle over a space  $X$  with trivial involution  $\tau = \text{Id}_X$ . Then, the associated determinant line bundle  $\det(\mathcal{E})$  endowed with the “Real” structure  $\det(\Theta)$  is  $\mathfrak{R}$ -trivial and admits a unique canonical  $\mathfrak{R}$ -section  $s : X \rightarrow \mathbb{S}(\det(\mathcal{E}))$ .*

If  $X^\tau \neq \emptyset$  the restricted vector bundle  $\mathcal{E}|_{X^\tau} \rightarrow X^\tau$  is a  $\mathfrak{Q}$ -bundle over a space with trivial involution. Proposition 2.5 assures that the restricted line bundle  $\det(\mathcal{E}|_{X^\tau})$  is  $\mathfrak{R}$ -trivial with respect to the restricted “Real” structure  $\det(\Theta|_{X^\tau})$  and admits a distinguished  $\mathfrak{R}$ -section

$$s_{\mathcal{E}} : X^\tau \rightarrow \mathbb{S}(\det(\mathcal{E})|_{X^\tau}). \quad (2.3)$$

We will refer to  $s_{\mathcal{E}}$  as the *canonical section* associated to  $(\mathcal{E}, \Theta)$ .

**2.3. A short reminder of the equivariant Borel cohomology.** The proper cohomology theory for the study of vector bundles in the category of spaces with involution is the equivariant cohomology introduced by A. Borel in [Bo]. This cohomology has been used for the topological classification of “Real” vector bundles [DG1] and plays also a role in the classification of “Quaternionic” vector bundles [DG2]. A short self-consistent summary of this cohomology theory can be found in [DG1, Section 5.1] and we refer to [Hs, Chapter 3] and [AP, Chapter 1] for a more complete introduction to the subject.

Since we need this tool we briefly recall the main steps of the Borel construction. The *homotopy quotient* of an involutive space  $(X, \tau)$  is the orbit space

$$X_{\sim\tau} := X \times \mathbb{S}^\infty / (\tau \times \theta_\infty). \quad (2.4)$$

Here  $\theta_\infty$  is the antipodal map on the infinite sphere  $\mathbb{S}^\infty$  (cf. [DG1, Example 4.1]) and  $\mathbb{S}^{0,\infty}$  is used as short notation for the pair  $(\mathbb{S}^\infty, \theta_\infty)$ . The product space  $X \times \mathbb{S}^\infty$  (forgetting for a moment the  $\mathbb{Z}_2$ -action) has the *same* homotopy type of  $X$  since  $\mathbb{S}^\infty$  is contractible. Moreover, since  $\theta_\infty$  is a free involution, also the composed involution  $\tau \times \theta_\infty$  is free, independently of  $\tau$ . Let  $\mathcal{R}$  be any commutative ring (e. g.,  $\mathbb{R}, \mathbb{Z}, \mathbb{Z}_2, \dots$ ). The *equivariant* cohomology ring of  $(X, \tau)$  with coefficients in  $\mathcal{R}$  is defined as

$$H_{\mathbb{Z}_2}^\bullet(X, \mathcal{R}) := H^\bullet(X_{\sim\tau}, \mathcal{R}).$$

More precisely, each equivariant cohomology group  $H_{\mathbb{Z}_2}^j(X, \mathcal{R})$  is given by the singular cohomology group  $H^j(X_{\sim\tau}, \mathcal{R})$  of the homotopy quotient  $X_{\sim\tau}$  with coefficients in  $\mathcal{R}$  and the ring structure is given, as usual, by the cup product. As the coefficients of the usual singular cohomology are generalized to *local coefficients* (see e. g. [Hat, Section 3.H] or [DK, Section 5]), the coefficients of the Borel equivariant cohomology are also generalized to local coefficients. Given an involutive space  $(X, \tau)$  one can consider the homotopy group  $\pi_1(X_{\sim\tau})$  and the associated *group ring*  $\mathbb{Z}[\pi_1(X_{\sim\tau})]$ . Each module  $\mathcal{Z}$  over the group  $\mathbb{Z}[\pi_1(X_{\sim\tau})]$  is, by definition, a *local system* on  $X_{\sim\tau}$ . Using this local system one defines, as usual, the equivariant cohomology with local coefficients in  $\mathcal{Z}$ :

$$H_{\mathbb{Z}_2}^\bullet(X, \mathcal{Z}) := H^\bullet(X_{\sim\tau}, \mathcal{Z}).$$

We are particularly interested in modules  $\mathcal{Z}$  whose underlying groups are identifiable with  $\mathbb{Z}$ . For each involutive space  $(X, \tau)$ , there always exists a particular family of local systems  $\mathbb{Z}(j)$  labelled by  $j \in \mathbb{Z}$ . Here  $\mathbb{Z}(j) \simeq X \times \mathbb{Z}$  denotes the  $\mathbb{Z}_2$ -equivariant local system on  $(X, \tau)$  made equivariant by the  $\mathbb{Z}_2$ -action  $(x, l) \mapsto (\tau(x), (-1)^j l)$ . Given that the module structure depends only on the parity of  $j$ , one can consider only the  $\mathbb{Z}_2$ -modules  $\mathbb{Z}(0)$  and  $\mathbb{Z}(1)$ . Noticing that  $\mathbb{Z}(0)$  corresponds to the case of the trivial action of  $\pi_1(X_{\sim\tau})$  on  $\mathbb{Z}$  one has that  $H_{\mathbb{Z}_2}^k(X, \mathbb{Z}(0)) \simeq H_{\mathbb{Z}_2}^k(X, \mathbb{Z})$  [DK, Section 5.2].

We recall the two important group isomorphisms

$$\begin{aligned} H_{\mathbb{Z}_2}^1(X, \mathbb{Z}(1)) &\simeq [X, \mathbb{U}(1)]_{\mathbb{Z}_2} \simeq [X, \mathbb{S}^{1,1}]_{\mathbb{Z}_2} \\ H_{\mathbb{Z}_2}^2(X, \mathbb{Z}(1)) &\simeq \text{Vec}_{\mathfrak{R}}^1(X, \tau) \equiv \text{Pic}_{\mathfrak{R}}(X, \tau) \end{aligned} \quad (2.5)$$

involving the first two equivariant cohomology groups. The first isomorphism [Go, Proposition A.2] says that the first equivariant cohomology group is isomorphic to the set of  $\mathbb{Z}_2$ -homotopy classes of equivariant maps  $\varphi : X \rightarrow \mathbb{U}(1)$  where the involution on  $\mathbb{U}(1)$  is induced by the complex conjugation, i. e.  $\varphi(\tau(x)) = \overline{\varphi(x)}$ . Of course there is a natural identification of this target involutive space with the involutive sphere  $\mathbb{S}^{1,1}$ . The second isomorphism is due to B. Kahn [Kah] and expresses the equivalence between the Picard group of “Real” line bundles (in the sense of [At1, DG1]) over  $(X, \tau)$  and the second equivariant cohomology group of this space. A more modern proof of this isomorphism can be found in [Go, Corollary A.5].

The fixed point subset  $X^\tau \subset X$  is closed and  $\tau$ -invariant and the inclusion  $\iota : X^\tau \hookrightarrow X$  extends to an inclusion  $\iota : X_{\sim\tau}^\tau \hookrightarrow X_{\sim\tau}$  of the respective homotopy quotients. The *relative* equivariant cohomology can be defined as usual by the identification

$$H_{\mathbb{Z}_2}^\bullet(X|X^\tau, \mathcal{Z}) := H^\bullet(X_{\sim\tau}|X_{\sim\tau}^\tau, \mathcal{Z}).$$

Consequently, one has the related long exact sequence in cohomology

$$\dots \longrightarrow H_{\mathbb{Z}_2}^j(X|X^\tau, \mathcal{Z}) \xrightarrow{\delta_2} H_{\mathbb{Z}_2}^j(X, \mathcal{Z}) \xrightarrow{r} H_{\mathbb{Z}_2}^j(X^\tau, \mathcal{Z}) \xrightarrow{\delta_1} H_{\mathbb{Z}_2}^{j+1}(X|X^\tau, \mathcal{Z}) \longrightarrow \dots \quad (2.6)$$

where the map  $r := \iota^*$  restricts cochains on  $X_{\sim\tau}$  to cochains on  $X_{\sim\tau}^\tau$ . The  $j$ -th *cokernel* of  $r$  is by definition

$$\text{Coker}^j(X|X^\tau, \mathcal{Z}) := H_{\mathbb{Z}_2}^j(X^\tau, \mathcal{Z}) / r(H_{\mathbb{Z}_2}^j(X, \mathcal{Z})). \quad (2.7)$$

Let us point out that with the same construction one can define relative cohomology theories  $H_{\mathbb{Z}_2}^\bullet(X|Y, \mathcal{Z})$  for each closed subset  $Y \subseteq X$  which is  $\tau$ -invariant  $\tau(Y) = Y$ . If  $Y = \emptyset$  then  $H_{\mathbb{Z}_2}^j(X|\emptyset, \mathcal{Z}) \simeq H_{\mathbb{Z}_2}^j(X, \mathcal{Z})$  by definition. Therefore it is suitable to declare  $H_{\mathbb{Z}_2}^j(\emptyset, \mathcal{Z}) = 0$  for consistency with the exact sequence (2.6).

**2.4. A geometric model for the relative equivariant cohomology.** In this section we provide a geometric model for the equivariant relative cohomology group  $H_{\mathbb{Z}_2}^2(X|Y, \mathbb{Z}(1))$  which extends the second isomorphism in (2.5).

Let  $(X, \tau)$  be an involutive space and  $Y \subseteq X$  a closed  $\tau$ -invariant subspace. Consider pairs  $(\mathcal{L}, s)$  consisting of: (I) a “Real” line bundle  $\mathcal{L} \rightarrow X$  with a given “Real” structure  $\Theta$  and a Hermitian metric  $\mathfrak{m}$ ; (II) a (nowhere vanishing) “Real” section  $s : Y \rightarrow \mathbb{S}(\mathcal{L}|_Y)$  of the circle bundle associated to the restriction  $\mathcal{L}|_Y \rightarrow Y$ . Two pairs  $(\mathcal{L}_1, s_1)$  and  $(\mathcal{L}_2, s_2)$  builded over the same involutive base space  $(X, \tau)$  and same  $\tau$ -invariant subspace  $Y \subseteq X$  are said *isomorphic* if there is an  $\mathfrak{R}$ -isomorphism  $f : \mathcal{L}_1 \rightarrow \mathcal{L}_2$  (preserving the Hermitian structure) such that  $f \circ s_1 = s_2$ .

**Definition 2.6** (Relative “Real” Picard group). *For a given involutive space  $(X, \tau)$  and a closed subspace  $Y \subseteq X$  such that  $\tau(Y) = Y$ , we define  $\text{Pic}_{\mathfrak{R}}(X|Y, \tau)$  to be the group of the isomorphism classes of pairs  $(\mathcal{L}, s)$ , with group structure given by the tensor product*

$$(\mathcal{L}_1, s_1) \otimes (\mathcal{L}_2, s_2) \simeq (\mathcal{L}_1 \otimes \mathcal{L}_2, s_1 \otimes s_2).$$

The main result of this section is the following characterization.

**Proposition 2.7.** *There is a natural isomorphism of groups*

$$\tilde{\kappa} : \text{Pic}_{\mathfrak{R}}(X|Y, \tau) \xrightarrow{\simeq} H_{\mathbb{Z}_2}^2(X|Y, \mathbb{Z}(1)).$$

*Proof (sketch of).* The Kahn’s isomorphism  $H_{\mathbb{Z}_2}^2(X, \mathbb{Z}(1)) \simeq \text{Pic}_{\mathfrak{R}}(X, \tau)$  can be proved roughly in two steps: The first step is to prove that  $\text{Vec}_{\mathfrak{R}}^1(X, \tau)$  is isomorphic to the sheaf cohomology  $H^1(\mathbb{Z}_2^\bullet \times X, \underline{\mathbb{S}}_\bullet^1)$  of the simplicial space  $\mathbb{Z}_2^\bullet \times X$  associated to the involutive space  $(X, \tau)$  and this is carried out by realizing the sheaf cohomology as a Čech cohomology; The second step is to identify the sheaf cohomology with  $H_{\mathbb{Z}_2}^2(X, \mathbb{Z}(1))$ . For the full argument we refer to [Go, Appendix A]. The same argument is replicable verbatim for the pair  $Y \subseteq X$ . Firstly, the use of the Čech cohomology allows to prove the group isomorphism between  $\text{Pic}_{\mathfrak{R}}(X|Y, \tau)$  and the relative sheaf cohomology  $H^1(\mathbb{Z}_2^\bullet \times X|\mathbb{Z}_2^\bullet \times Y, \underline{\mathbb{S}}_\bullet^1)$ . Secondly, this sheaf cohomology can be identified with the relative equivariant cohomology  $H^2(X|Y, \mathbb{Z}(1))$  in the same way as in [Go, Appendix A]. ■

In the next sections we will exploit the result of Proposition 2.7 in the special case of  $Y \subseteq X^\tau$ . We recall that in this situation, as a consequence of Proposition 2.5, the choice of  $s$  is *canonical* and corresponds to a trivialization of the restricted “Real” line bundle  $\mathcal{L}|_Y \rightarrow Y$ .

**2.5. Generalized FKMM-invariant.** By combining the content of Proposition 2.5 with that of Proposition 2.7 we can introduce an *intrinsic* invariant for the category of “Quaternionic” vector bundles.

**Definition 2.8** (Generalized FKMM-invariant). *Let  $(\mathcal{E}, \Theta)$  be an even rank “Quaternionic” vector bundle over the involutive space  $(X, \tau)$  and consider the pair  $(\det(\mathcal{E}), s_{\mathcal{E}})$  where  $\det(\mathcal{E})$  is the determinant line bundle associated to  $\mathcal{E}$  endowed with the “Real” structure induced by  $\det(\Theta)$  and  $s_{\mathcal{E}}$  is the*

canonical section described by (2.3). The generalized FKMM-invariant of  $(\mathcal{E}, \Theta)$  is the cohomology class  $\kappa(\mathcal{E}, \Theta) \in H_{\mathbb{Z}_2}^2(X|X^\tau, \mathbb{Z}(1))$  given by

$$\kappa(\mathcal{E}, \Theta) := \tilde{\kappa}([\det(\mathcal{E}), s_{\mathcal{E}}])$$

where  $[(\det(\mathcal{E}), s_{\mathcal{E}})] \in \text{Pic}_{\mathfrak{R}}(X|X^\tau, \tau)$  is the isomorphism class of the pair  $(\det(\mathcal{E}), s_{\mathcal{E}})$  and  $\tilde{\kappa}$  is the group isomorphism described in Proposition 2.7.

The following properties are immediate consequence of the last definition.

- (1) Isomorphic “Quaternionic” vector bundles define the same FKMM-invariant;
- (2) The FKMM-invariant is *natural* under the pullback induced by equivariant maps;
- (3) If  $(\mathcal{E}, \Theta)$  is  $\mathfrak{Q}$ -trivial then  $\kappa(\mathcal{E}, \Theta) = 0$ ;
- (4) The FKMM-invariant is *additive* with respect to the Whitney sum and the abelian structure of  $H_{\mathbb{Z}_2}^2(X|X^\tau, \mathbb{Z}(1))$ , namely

$$\kappa(\mathcal{E}_1 \oplus \mathcal{E}_2, \Theta_1 \oplus \Theta_2) = \kappa(\mathcal{E}_1, \Theta_1) \cdot \kappa(\mathcal{E}_2, \Theta_2)$$

for each pair of “Quaternionic” vector bundles  $(\mathcal{E}_1, \Theta_1)$  and  $(\mathcal{E}_2, \Theta_2)$  over the same involutive space  $(X, \tau)$ .

**Remark 2.9** (Comparison with the original definition of the FKMM-invariant). The original definition of the FKMM-invariant, firstly introduced in [FKMM] and then extensively used in [DG2], is strongly based on two crucial assumptions: (a)  $X^\tau \neq \emptyset$  and (b)  $\det(\mathcal{E})$  has to be  $\mathfrak{R}$ -trivial (cf. [DG2, Definition 3.1]). Under this assumption the FKMM-invariant can be defined as an element in the cokernel of the restriction map  $r : H_{\mathbb{Z}_2}^1(X, \mathbb{Z}(1)) \rightarrow H_{\mathbb{Z}_2}^1(X^\tau, \mathbb{Z}(1))$ . Moreover, the isomorphism described in [DG2, Lemma 3.1] allows to describe the original FKMM-invariant as an equivariant map  $\phi : X^\tau \rightarrow \mathbb{U}(1)$  which measures the difference between the canonical section  $s_{\mathcal{E}}$  and a global “Real” section of  $\det(\mathcal{E})$  (cf. [DG2, Definition 3.2 & Remark 3.2]). On the other hand, the generalized invariant introduced in Definition 2.8 does not require any extra restrictions on the nature of  $(X, \tau)$  and  $(\mathcal{E}, \Theta)$  and fulfills all the structural properties of the original FKMM-invariant (compare the (1)-(4) above with the content of [DG2, Theorem 3.1]). Moreover, as soon as assumptions (a) and (b) are met, the generalized FKMM-invariant agrees with the original FKMM-invariant (see Proposition 2.10 and Corollary 2.11 below). This allows to state that Definition 2.8 really *extends* the notion of the FKMM-invariant. ◀

Let  $\text{Coker}^1(X|X^\tau, \mathbb{Z}(1))$  be the cokernel described in (2.7) and  $[X, \mathbb{U}(1)]_{\mathbb{Z}_2}$  the set of  $\mathbb{Z}_2$ -homotopy classes of equivariant maps between the involutive space  $(X, \tau)$  and the group  $\mathbb{U}(1)$  endowed with the involution induced by the complex conjugation. Let us recall that the isomorphism

$$[X^\tau, \mathbb{U}(1)]_{\mathbb{Z}_2} / [X, \mathbb{U}(1)]_{\mathbb{Z}_2} \simeq \text{Coker}^1(X|X^\tau, \mathbb{Z}(1))$$

holds under the assumption  $X^\tau \neq \emptyset$  [DG2, Lemma 3.1].

**Proposition 2.10.** *Let  $(\mathcal{E}, \Theta)$  be an even rank “Quaternionic” vector bundle over the involutive space  $(X, \tau)$ . Assume that: (a)  $X^\tau \neq \emptyset$  and (b)  $\det(\mathcal{E})$  is  $\mathfrak{R}$ -trivial. Then generalized FKMM-invariant  $\kappa(\mathcal{E}, \Theta)$  introduced in Definition 2.8 is the injective image of an element  $[\phi] \in \text{Coker}^1(X|X^\tau, \mathbb{Z}(1))$ .*

*Proof.* The homomorphism  $\delta_1$  and  $\delta_2$  in the long exact sequence

$$\dots \rightarrow H_{\mathbb{Z}_2}^1(X, \mathbb{Z}(1)) \xrightarrow{r} H_{\mathbb{Z}_2}^1(X^\tau, \mathbb{Z}(1)) \xrightarrow{\delta_1} H_{\mathbb{Z}_2}^2(X|X^\tau, \mathbb{Z}(1)) \xrightarrow{\delta_2} H_{\mathbb{Z}_2}^2(X, \mathbb{Z}(1)) \rightarrow \dots \quad (2.8)$$

for the pair  $X^\tau \hookrightarrow X$  can be interpreted as homomorphisms

$$\delta_1 : [X^\tau, \mathbb{U}(1)]_{\mathbb{Z}_2} \rightarrow \text{Pic}_{\mathfrak{R}}(X|X^\tau, \tau), \quad \delta_2 : \text{Pic}_{\mathfrak{R}}(X|X^\tau, \tau) \rightarrow \text{Pic}_{\mathfrak{R}}(X, \tau)$$

in view of the isomorphism proved in Proposition 2.7 and the isomorphisms in (2.5), respectively. Moreover, the triviality of  $\det(\mathcal{E})$  implies that the pair  $(\det(\mathcal{E}), s_{\mathcal{E}})$  which enters into the definition of the FKMM-invariant is an element of the class  $[(X \times \mathbb{U}(1), \phi)] \in \text{Pic}_{\mathfrak{R}}(X|X^\tau, \tau)$  with  $\phi : X^\tau \rightarrow \mathbb{U}(1)$  is

a suitable equivariant map. Therefore  $\delta_1$  turns out to be just the assignment of the class  $[(X \times \mathbb{U}(1), \phi)]$  to the map  $[\phi] \in [X^\tau, \mathbb{U}(1)]_{\mathbb{Z}_2}$  and  $\delta_2$  agrees with the map induced from  $(\mathcal{L}, s) \mapsto \mathcal{L}$ . With this interpretation the result directly follows from the exact sequence. Finally, the injectivity is assured by

$$0 \longrightarrow \text{Coker}^1(X|X^\tau, \mathbb{Z}(1)) \xrightarrow{\delta_1} H_{\mathbb{Z}_2}^2(X|X^\tau, \mathbb{Z}(1)) \xrightarrow{\delta_2} H_{\mathbb{Z}_2}^2(X, \mathbb{Z}(1)) \longrightarrow \dots$$

which follows from (2.8) by general arguments.  $\blacksquare$

As it emerges from the proof, the element  $[\phi]$  in the statement of Proposition 2.10 is represented by the canonical section  $s_{\mathcal{E}}$  modulo the action (multiplication and restriction) of an equivariant map  $s : X \rightarrow \mathbb{U}(1)$  which can be interpreted as a global trivialization of the trivial line bundle  $\det(\mathcal{E})$ . Therefore  $[\phi]$  is just the original FKMM-invariant as introduced in [FKMM] and [DG2, Definition 3.2]. As a consequence, we can rephrase Proposition 2.10 by saying that:

*“The generalized FKMM-invariant is just the image (under  $\delta_1$ ) of the old FKMM-invariant whenever the latter can be defined.”*

According to [DG2, Definition 1.1] an FKMM-space is an involutive space  $(X, \tau)$  such that  $X^\tau \neq \emptyset$  and  $H_{\mathbb{Z}_2}^2(X, \mathbb{Z}(1)) = 0$ .

**Corollary 2.11.** *Let  $(\mathcal{E}, \Theta)$  be an even rank “Quaternionic” vector bundle over the FKMM-space  $(X, \tau)$ . Then, the generalized FKMM-invariant  $\kappa(\mathcal{E}, \Theta)$  introduced in Definition 2.8 is represented by a map  $\phi : X^\tau \rightarrow \{+1, -1\}$ .*

*Proof.* It has been proved in [DG2, Lemma 3.1] that the condition  $H_{\mathbb{Z}_2}^2(X, \mathbb{Z}(1)) = 0$  implies that  $\delta_1$  defines an isomorphism between  $\text{Coker}^1(X|X^\tau, \mathbb{Z}(1))$  and  $H_{\mathbb{Z}_2}^2(X|X^\tau, \mathbb{Z}(1))$ . Hence,  $\kappa(\mathcal{E}, \Theta)$  can be viewed as an element in  $\text{Coker}^1(X|X^\tau, \mathbb{Z}(1))$  which, modulo equivalences, is represented by an equivariant function  $\phi : X^\tau \rightarrow \mathbb{U}(1)$ . Since, by definition,  $\tau$  acts trivially on  $X^\tau$ ,  $\phi$  maps on the point of  $\mathbb{U}(1)$  fixed by the complex conjugation.  $\blacksquare$

For the next result we need to recall that the Kahn’s isomorphism  $H_{\mathbb{Z}_2}^2(X, \mathbb{Z}(1)) \simeq \text{Pic}_{\mathbb{R}}(X, \tau)$  is realized by first “Real” Chern class  $c_1^{\mathbb{R}}$  (see [Kah] or [DG1, Section 5.2]).

**Corollary 2.12.** *Let  $(\mathcal{E}, \Theta)$  be an even rank “Quaternionic” vector bundle over the involutive space  $(X, \tau)$  and*

$$\delta_2 : H_{\mathbb{Z}_2}^2(X|X^\tau, \mathbb{Z}(1)) \longrightarrow H_{\mathbb{Z}_2}^2(X, \mathbb{Z}(1))$$

*the homomorphism in the long exact sequence (2.8) for the pair  $X^\tau \hookrightarrow X$ . Then:*

- (1) *The homomorphism  $\delta_2$  carries the FKMM-invariant  $\kappa(\mathcal{E}, \Theta)$  to the first “Real” Chern class  $c_1^{\mathbb{R}}(\det(\mathcal{E}))$  of the “Real” line bundle  $\det(\mathcal{E}) \rightarrow X$ ;*
- (2) *If in addition  $X^\tau = \emptyset$ , then the FKMM-invariant  $\kappa(\mathcal{E}, \Theta)$  agrees with  $c_1^{\mathbb{R}}(\det(\mathcal{E}))$ .*

*Proof.* From the proof of Proposition 2.10 one knows that the  $\delta_2$  can be interpreted as the homomorphism  $\text{Pic}_{\mathbb{R}}(X|X^\tau, \tau) \rightarrow \text{Pic}_{\mathbb{R}}(X, \tau)$  induced by  $(\mathcal{L}, s) \mapsto \mathcal{L}$ . Therefore (1) follows from the fact that the Kahn’s isomorphism is induced by  $c_1^{\mathbb{R}}$ . (2) follows just by observing that the condition  $X^\tau = \emptyset$ , once inserted in the long exact sequence (2.8), forces  $\delta_2$  to be an isomorphism.  $\blacksquare$

The original FKMM-invariant cannot be defined when  $X^\tau = \emptyset$  but, on the contrary, requires  $X^\tau \neq \emptyset$  and the triviality of  $\det(\mathcal{E})$ . On the other hand the generalized FKMM-invariant introduced in Definition 2.8 bypasses these restrictions and provides a way to define a topological invariant also in the *opposite* case when  $X^\tau = \emptyset$  and  $\det(\mathcal{E})$  is non-trivial. In contrast to the definition of FKMM-space [DG2, Definition 1.1] we can call “anti-FKMM” an involutive space  $(X, \tau)$  such that  $X^\tau = \emptyset$  and  $H_{\mathbb{Z}_2}^2(X, \mathbb{Z}(1)) \neq 0$ . Corollary 2.11 says that the generalized FKMM-invariant agrees with the original invariant in the case of an FKMM-space and Corollary 2.12 (ii) shows that the generalized FKMM-invariant reduces to the first “Real” Chern class in the anti-FKMM case. In more general situations the classification of even rank “Quaternionic” vector bundles would require the generalized notion of FKMM-invariant (in the form of Definition 2.8) which can be considered a sort of *mixture* of the two mentioned extreme cases.



**2.6. Universal FKMM-invariant.** Even rank “Quaternionic” vector bundles can be classified by maps with values in a classification space [DG2, Theorem 2.4]. This is the basic fact which allows a description of the generalized FKMM-invariant as a *characteristic class*, namely as the image of a unique universal object under the pullback induced by the classification maps. This section provides a clarification and a simplification of some notions and results already discussed in [DG2, Section 6].

Let us recall that the (even dimensional) *Grassmannian* is defined as

$$G_{2m}(\mathbb{C}^\infty) := \bigcup_{n=2m}^{\infty} G_{2m}(\mathbb{C}^n),$$

where, for each pair  $2m \leq n$ ,  $G_{2m}(\mathbb{C}^n) \simeq \mathbb{U}(n)/(\mathbb{U}(2m) \times \mathbb{U}(n-2m))$  is the set of  $2m$ -dimensional (complex) subspaces of  $\mathbb{C}^n$ . The spaces  $G_{2m}(\mathbb{C}^n)$  are finite CW-complexes and the space  $G_{2m}(\mathbb{C}^\infty)$  inherits the direct limit topology given by the inclusions  $G_m(\mathbb{C}^n) \hookrightarrow G_m(\mathbb{C}^{n+1}) \hookrightarrow \dots$  induced by  $\mathbb{C}^n \subset \mathbb{C}^{n+1} \subset \dots$ . Moreover,  $G_{2m}(\mathbb{C}^\infty)$  can be endowed with an involution of quaternionic type in the following way: Let  $\Sigma = \langle v_1, v_2, \dots, v_{2m-1}, v_{2m} \rangle_{\mathbb{C}}$  be any  $2m$ -plane in  $G_{2m}(\mathbb{C}^{2n})$  generated by the basis  $\{v_1, v_2, \dots, v_{2m-1}, v_{2m}\}$  and define a new point  $\rho(\Sigma) \in G_{2m}(\mathbb{C}^{2n})$  as the  $2m$ -plane spanned by  $\langle Q\bar{v}_1, Q\bar{v}_2, \dots, Q\bar{v}_{2m-1}, Q\bar{v}_{2m} \rangle_{\mathbb{C}}$  where  $\bar{v}_j$  is the complex conjugate of  $v_j$  and  $Q$  is the  $2n \times 2n$  matrix (2.1). Notice that the definition of  $\rho(\Sigma)$  does not depend on the choice of a particular basis and the map  $\rho : G_{2m}(\mathbb{C}^n) \rightarrow G_{2m}(\mathbb{C}^n)$  is an involution that makes the pair  $(G_{2m}(\mathbb{C}^{2n}), \rho)$  into an involutive space. Since all the inclusions  $G_{2m}(\mathbb{C}^{2n}) \hookrightarrow G_{2m}(\mathbb{C}^{2(n+1)}) \hookrightarrow \dots$  are equivariant, the involution extends to the infinite Grassmannian in such a way that  $\hat{G}_{2m}(\mathbb{C}^\infty) \equiv (G_{2m}(\mathbb{C}^\infty), \rho)$  becomes an involutive space. Let  $\Sigma = \rho(\Sigma)$  be a fixed point of  $\hat{G}_{2m}(\mathbb{C}^\infty)$ . Since  $\rho$  acts on vectors as a quaternionic structure one has  $\Sigma \simeq \mathbb{H}^m$  (cf. [DG2, Remark 2.1]). This fact implies that the fixed point set of  $\hat{G}_{2m}(\mathbb{C}^\infty)$  can be identified with the *quaternionic* Grassmannian, namely  $\hat{G}_{2m}(\mathbb{C}^\infty)^\rho \simeq G_m(\mathbb{H}^\infty)$  (see [DG2, Section 2.4] for more details).

Each manifold  $G_m(\mathbb{C}^n)$  is the base space of a canonical rank  $m$  complex vector bundle  $\pi : \mathcal{T}_m^n \rightarrow G_m(\mathbb{C}^n)$  with total space  $\mathcal{T}_m^n := \{(\Sigma, v) \in (G_m(\mathbb{C}^n) \times \mathbb{C}^n) \mid v \in \Sigma\}$  and bundle projection  $\pi(\Sigma, v) = \Sigma$ . When  $n$  tends to infinity, the same construction leads to the *tautological  $m$ -plane bundle*  $\pi : \mathcal{T}_m^\infty \rightarrow G_m(\mathbb{C}^\infty)$ . The latter is the universal vector bundle which classifies complex vector bundles. In fact any rank  $m$  complex vector bundle  $\mathcal{E} \rightarrow X$  is realized, up to isomorphisms, as a pullback  $\mathcal{E} \simeq \varphi^* \mathcal{T}_m^\infty$  with respect to a *classifying map*  $\varphi : X \rightarrow G_m(\mathbb{C}^\infty)$ . Since pullbacks of homotopy equivalent maps yield isomorphic vector bundles one gets the well known result  $\text{Vec}_{\mathbb{C}}^m(X) \simeq [X, G_m(\mathbb{C}^\infty)]$ . This result extends to the category of even rank “Quaternionic” vector bundles provided that the total space  $\mathcal{T}_{2m}^\infty$  is endowed with a  $\mathfrak{Q}$ -structure compatible with the involution  $\rho$ . This is done by the anti-linear map  $\Xi : \mathcal{T}_{2m}^\infty \rightarrow \mathcal{T}_{2m}^\infty$  defined by  $\Xi : (\Sigma, v) \mapsto (\rho(\Sigma), Q\bar{v})$ . The relation  $\pi \circ \Xi = \rho \circ \pi$  assures that  $(\mathcal{T}_{2m}^\infty, \Xi)$  is a  $\mathfrak{Q}$ -bundle over the involutive space  $\hat{G}_{2m}(\mathbb{C}^\infty)$ . This *tautological  $\mathfrak{Q}$ -bundle* classifies “Quaternionic” vector bundles in the sense that  $\text{Vec}_{\mathfrak{Q}}^{2m}(X, \tau) \simeq [X, \hat{G}_{2m}(\mathbb{C}^\infty)]_{\mathbb{Z}_2}$  where in the right-hand side there is the set of  $\mathbb{Z}_2$ -homotopy classes of equivariant maps between  $(X, \tau)$  and  $\hat{G}_{2m}(\mathbb{C}^\infty)$  [DG2, Theorem 2.4].

The construction of the generalized FKMM-invariant applies to the universal  $\mathfrak{Q}$ -bundle  $(\mathcal{T}_{2m}^\infty, \Xi)$ .

**Definition 2.13** (Universal FKMM-invariant). *The universal FKMM-invariant*

$$\mathfrak{h}_{\text{univ}} \in H_{\mathbb{Z}_2}^2(\hat{G}_{2m}(\mathbb{C}^\infty) | \hat{G}_{2m}(\mathbb{C}^\infty)^\rho, \mathbb{Z}(1))$$

is the (generalized) FKMM-invariant of the tautological  $\mathfrak{Q}$ -bundle  $(\mathcal{T}_{2m}^\infty, \Xi)$  as constructed in Definition 2.8. More precisely  $\mathfrak{h}_{\text{univ}} := \kappa(\mathcal{T}_{2m}^\infty, \Xi)$  is the image of

$$[(\det(\mathcal{T}_{2m}^\infty), s_{\mathcal{T}_{2m}^\infty})] \in \text{Vec}_{\mathfrak{R}}^1(\hat{G}_{2m}(\mathbb{C}^\infty) | \hat{G}_{2m}(\mathbb{C}^\infty)^\rho, \rho)$$

under the natural isomorphism  $\tilde{\kappa}$  described in Proposition 2.7.

The *naturality* of the invariant  $\kappa$  implies the following important result.

**Theorem 2.14** (Universality). *Let  $(\mathcal{E}, \Theta)$  be a rank  $2m$  “Quaternionic” vector bundle over the involutive space  $(X, \tau)$  classified by the equivariant map  $\varphi : X \rightarrow \hat{G}_{2m}(\mathbb{C}^\infty)$ . The generalized FKMM-invariant of  $(\mathcal{E}, \Theta)$  verifies*

$$\kappa(\mathcal{E}, \Theta) = \varphi^*(\mathfrak{h}_{\text{univ}})$$

where  $\varphi^*$  is the homomorphism in cohomology induced by  $\varphi$ .

The identification of  $\mathfrak{h}_{\text{univ}}$  as an element of  $H_{\mathbb{Z}_2}^2(\hat{G}_{2m}(\mathbb{C}^\infty)|\hat{G}_{2m}(\mathbb{C}^\infty)^\rho, \mathbb{Z}(1))$  requires a careful investigation of the equivariant cohomology of  $\hat{G}_{2m}(\mathbb{C}^\infty)$ . Some aspects of this analysis are quite technical and are postponed in Appendix A. The main results are summarized below.

**Proposition 2.15** (Identification). *The following facts hold:*

- (1) *The group  $H_{\mathbb{Z}_2}^2(\hat{G}_{2m}(\mathbb{C}^\infty), \mathbb{Z}(1)) \simeq \mathbb{Z}$  is generated by the (universal) first “Real” Chern class*

$$\mathfrak{c}_1^{\mathfrak{R}} := \mathfrak{c}_1^{\mathfrak{R}}(\det(\mathcal{T}_{2m}^\infty))$$

*of the “Real” line bundle  $\det(\mathcal{T}_{2m}^\infty) \rightarrow \hat{G}_{2m}(\mathbb{C}^\infty)$  associated to the tautological  $\mathfrak{Q}$ -bundle  $(\mathcal{T}_{2m}^\infty, \Xi)$ . Moreover, under the map  $f : H_{\mathbb{Z}_2}^2(\hat{G}_{2m}(\mathbb{C}^\infty), \mathbb{Z}(1)) \rightarrow H^2(G_{2m}(\mathbb{C}^\infty), \mathbb{Z})$  which forgets the  $\mathbb{Z}_2$ -action the class  $\mathfrak{c}_1^{\mathfrak{R}}$  is mapped in the first universal Chern class  $\mathfrak{c}_1 \in H^2(G_{2m}(\mathbb{C}^\infty), \mathbb{Z})$ ;*

- (2) *The homomorphism*

$$\delta_2 : H_{\mathbb{Z}_2}^2(\hat{G}_{2m}(\mathbb{C}^\infty)|\hat{G}_{2m}(\mathbb{C}^\infty)^\rho, \mathbb{Z}(1)) \xrightarrow{\simeq} H_{\mathbb{Z}_2}^2(\hat{G}_{2m}(\mathbb{C}^\infty)^\rho, \mathbb{Z}(1))$$

*in the long exact sequence (2.6) for the pair  $\hat{G}_{2m}(\mathbb{C}^\infty) \hookrightarrow \hat{G}_{2m}(\mathbb{C}^\infty)^\rho$  is, in fact, an isomorphism.*

- (3) *The universal FKMM-invariant  $\mathfrak{h}_{\text{univ}}$  is the generator of  $H_{\mathbb{Z}_2}^2(\hat{G}_{2m}(\mathbb{C}^\infty)|\hat{G}_{2m}(\mathbb{C}^\infty)^\rho, \mathbb{Z}(1)) \simeq \mathbb{Z}$  which is mapped by  $\delta_2$  in the first “Real” Chern class  $\mathfrak{c}_1^{\mathfrak{R}}$ , i. e.*

$$\mathfrak{h}_{\text{univ}} = \delta_2^{-1}(\mathfrak{c}_1^{\mathfrak{R}}).$$

*Proof.* The proof of (1) and (2) are given in Proposition A.3 and Proposition A.5, respectively. Item (3) follows from Corollary 2.12 (1).  $\blacksquare$

**Remark 2.16** (Comparison with the *old* definition of the universal FKMM-invariant). The *old* definition of the universal FKMM-invariant [DG2, Definition 6.1] is based on an intricate construction that we briefly recall. The bundle map  $\pi : \det(\mathcal{T}_{2m}^\infty) \rightarrow \hat{G}_{2m}(\mathbb{C}^\infty)$  defines an equivariant map between involutive spaces

$$\pi : (\mathbb{S}(\det(\mathcal{T}_{2m}^\infty)), \det(\Xi)) \longrightarrow \hat{G}_{2m}(\mathbb{C}^\infty).$$

The pullback of the tautological  $\mathfrak{Q}$ -bundle  $(\mathcal{T}_{2m}^\infty, \Xi)$  induced by  $\pi$  defines a  $\mathfrak{Q}$ -bundle  $(\pi^*\mathcal{T}_{2m}^\infty, \pi^*\Xi)$  over the involutive space  $(\mathbb{S}(\det(\mathcal{T}_{2m}^\infty)), \det(\Xi))$ . According to the old definition, the universal FKMM-invariant  $\mathfrak{R}_{\text{univ}}$  is the *old* FKMM-invariant (here denoted with  $\kappa_0$ ) of  $(\pi^*\mathcal{T}_{2m}^\infty, \pi^*\Xi)$ . More precisely

$$\mathfrak{R}_{\text{univ}} := \kappa_0(\pi^*\mathcal{T}_{2m}^\infty, \pi^*\Xi) \in \text{Coker}^1(\mathbb{S}(\det(\mathcal{T}_{2m}^\infty))|\mathbb{S}(\det(\mathcal{T}_{2m}^\infty))^\Xi, \mathbb{Z}(1)) \simeq \mathbb{Z}_2$$

agrees with the non trivial element of  $\mathbb{Z}_2$  [DG2, Theorem 6.2]. It is interesting to notice that the  $\mathfrak{Q}$ -line bundle  $(\pi^*\mathcal{T}_{2m}^\infty, \pi^*\Xi)$  is of FKMM-type [DG2, Lemma 6.2] and consequently the description in Proposition 2.10 applies. This means that the *new* FKMM-invariant

$$\kappa(\pi^*\mathcal{T}_{2m}^\infty, \pi^*\Xi) \in H_{\mathbb{Z}_2}^2(\mathbb{S}(\det(\mathcal{T}_{2m}^\infty))|\mathbb{S}(\det(\mathcal{T}_{2m}^\infty))^\Xi, \mathbb{Z}(1))$$

agree with the (non trivial) image of  $\kappa_0(\pi^*\mathcal{T}_{2m}^\infty, \pi^*\Xi)$  under the injective map  $\delta_1$ . Thus, one can replace in the definition of  $\mathfrak{R}_{\text{univ}}$  above the old invariant  $\kappa_0$  with the new invariant  $\kappa$ . Since  $\kappa$  is *natural* (by construction), it follows that

$$\mathfrak{R}_{\text{univ}} := \kappa(\pi^*\mathcal{T}_{2m}^\infty, \pi^*\Xi) = \pi^*\kappa(\mathcal{T}_{2m}^\infty, \Xi) = \pi^*(\mathfrak{h}_{\text{univ}}),$$

namely the old version of the universal FKMM-invariant is just the pullback of the new universal FKMM-invariant  $\mathfrak{h}_{\text{univ}}$  under the projection  $\pi$ .  $\blacktriangleleft$

### 3. “QUATERNIONIC” LINE BUNDLES AND FKMM-INVARIANT

As a matter of fact “Quaternionic” vector bundles of odd rank can be defined only over involutive base spaces  $(X, \tau)$  with a *free* involution (meaning that  $X^\tau = \emptyset$ ) [DG2, Proposition 2.1]. In this section we investigate the rank one case providing a classification scheme for “Quaternionic” *line* bundles. As a matter of notational simplification we prefer to replace in the next the pompous notation  $(\mathcal{L}, \Theta)$  with lighter symbols like  $\mathcal{L}_\mathbb{Q}$  and  $\mathcal{L}_\mathbb{R}$  for “Quaternionic” and “Real” line bundles, respectively.

**3.1. The “Quaternionic” Picard torsor.** We denote by  $\text{Pic}_\mathbb{R}(X, \tau) \equiv \text{Vec}_\mathbb{R}^1(X, \tau)$  and  $\text{Pic}_\mathbb{Q}(X, \tau) \equiv \text{Vec}_\mathbb{Q}^1(X, \tau)$  the sets of isomorphism classes of “Real” and “Quaternionic” line bundles on  $(X, \tau)$ , respectively. The set  $\text{Pic}_\mathbb{R}(X, \tau)$  gives rise to an abelian group under the tensor product of “Real” line bundles and it is known as the “Real” Picard group. On the other hand  $\text{Pic}_\mathbb{Q}(X, \tau)$  does not possess a group structure under the tensor product. Instead, under the essential assumption that  $\text{Pic}_\mathbb{Q}(X, \tau) \neq \emptyset$ , we can use the tensor product to define a left group-action of  $\text{Pic}_\mathbb{R}(X, \tau)$  on the set  $\text{Pic}_\mathbb{Q}(X, \tau)$ :

$$\begin{aligned} \text{Pic}_\mathbb{R}(X, \tau) \times \text{Pic}_\mathbb{Q}(X, \tau) &\longrightarrow \text{Pic}_\mathbb{Q}(X, \tau) \\ ([\mathcal{L}_\mathbb{R}], [\mathcal{L}_\mathbb{Q}]) &\longmapsto [\mathcal{L}_\mathbb{R} \otimes \mathcal{L}_\mathbb{Q}]. \end{aligned}$$

Let us introduce some terminology. For a group  $\mathbb{G}$ , a (left)  $\mathbb{G}$ -*tosor*, is a set  $T$  with a simply transitive (left)  $\mathbb{G}$ -action such that the map

$$\begin{aligned} \phi : \mathbb{G} \times T &\longrightarrow T \times T \\ (g, t) &\longmapsto (gt, t). \end{aligned}$$

is an isomorphism. In other words, the  $\mathbb{G}$ -action on  $T$  has to be free, and  $T$  has to be identifiable with a single  $\mathbb{G}$ -orbit. We notice that one may think of a  $\mathbb{G}$ -torsor as a principal  $\mathbb{G}$ -bundle over a single point.

**Theorem 3.1** (“Quaternionic” Picard torsor). *Assume  $\text{Pic}_\mathbb{Q}(X, \tau) \neq \emptyset$ . Then  $\text{Pic}_\mathbb{Q}(X, \tau)$  is a torsor under the group-action induced by  $\text{Pic}_\mathbb{R}(X, \tau)$ .*

*Proof.* Given a “Quaternionic” line bundle  $\mathcal{L}_\mathbb{Q}$  let  $\mathcal{L}_\mathbb{Q}^*$  be the associated *dual* bundle. The tensor product  $\mathcal{L}_\mathbb{Q} \otimes \mathcal{L}_\mathbb{Q}^*$  provides a representative for the trivial element in  $\text{Pic}_\mathbb{R}(X, \tau)$ . This fact can be checked, for instance, by looking at the transition functions. Given any pair  $\mathcal{L}_\mathbb{R}, \mathcal{L}'_\mathbb{R} \in \text{Pic}_\mathbb{R}(X, \tau)$  and a  $\mathcal{L}_\mathbb{Q} \in \text{Pic}_\mathbb{Q}(X, \tau)$  let assume that  $\mathcal{L}_\mathbb{R} \otimes \mathcal{L}_\mathbb{Q} \simeq \mathcal{L}'_\mathbb{R} \otimes \mathcal{L}_\mathbb{Q}$ . By tensorizing both sides with  $\mathcal{L}_\mathbb{Q}^*$  one obtains that  $\mathcal{L}_\mathbb{R} \simeq \mathcal{L}'_\mathbb{R}$ , namely  $\text{Pic}_\mathbb{R}(X, \tau)$  acts freely on  $\text{Pic}_\mathbb{Q}(X, \tau)$ . On the other hand, for each pair  $\mathcal{L}_\mathbb{Q}, \mathcal{L}'_\mathbb{Q} \in \text{Pic}_\mathbb{Q}(X, \tau)$  there is a “Real” line bundle  $\mathcal{L}_\mathbb{R} := \mathcal{L}'_\mathbb{Q} \otimes \mathcal{L}_\mathbb{Q}^*$  such that  $\mathcal{L}_\mathbb{R} \otimes \mathcal{L}_\mathbb{Q} \simeq \mathcal{L}'_\mathbb{Q}$ . Then the action of  $\text{Pic}_\mathbb{R}(X, \tau)$  is also transitive. ■

**Corollary 3.2** (Classification of “Quaternionic” line bundles). *Let  $(X, \tau)$  be an involutive space such that  $X^\tau = \emptyset$  and  $\text{Pic}_\mathbb{Q}(X, \tau) \neq \emptyset$ . Then,*

$$\text{Pic}_\mathbb{Q}(X, \tau) \simeq \text{Pic}_\mathbb{R}(X, \tau) \simeq H_{\mathbb{Z}_2}^2(X, \mathbb{Z}(1))$$

*is a bijection of sets.*

*Proof.* The first bijection is just a consequence of Theorem 3.1 which assures that  $\text{Pic}_\mathbb{Q}(X, \tau)$  can be realized as a single orbit under the left action of  $\text{Pic}_\mathbb{R}(X, \tau)$ . However, the form of the bijection is not natural but depends on the choice of an initial element in  $\text{Pic}_\mathbb{Q}(X, \tau)$ . The second bijection (in fact a group isomorphism) is proved in [Kah] or [Go, Corollary A.5]. ■

Theorem 3.1 provides a way to extend the notion of (generalized) FKMM-invariant for “Quaternionic” line bundles.

**Definition 3.3** (FKMM-invariant for “Quaternionic” line bundles). *Assume that  $\text{Pic}_\mathbb{Q}(X, \tau) \neq \emptyset$  and let us fix arbitrarily a reference element  $[\mathcal{L}_\text{ref}] \in \text{Pic}_\mathbb{Q}(X, \tau)$ . Given a  $\mathcal{L}_\mathbb{Q} \in \text{Pic}_\mathbb{Q}(X, \tau)$  let  $\mathcal{L}_\mathbb{R} \in \text{Pic}_\mathbb{R}(X, \tau)$  be the unique (up to isomorphisms) element such that  $[\mathcal{L}_\mathbb{Q}] = [\mathcal{L}_\mathbb{R} \otimes \mathcal{L}_\text{ref}]$ . The FKMM-invariant of  $\mathcal{L}_\mathbb{Q}$ , normalized by  $\mathcal{L}_\text{ref}$ , is by definition*

$$\kappa(\mathcal{L}_\mathbb{Q}) := c_1^\mathbb{R}(\mathcal{L}_\mathbb{R}).$$

Evidently the definition of the  $\kappa$ -invariant in the line bundle case suffers of the ambiguity due to the choice of a reference “Quaternionic” line bundle  $\mathcal{L}_{\text{ref}}$ . However, this ambiguity can be understood as the freedom to fix the normalization  $\kappa(\mathcal{L}_{\text{ref}}) = 0$ , a fact which is common in the theory of characteristic classes. From Definition 3.3 it results that  $\kappa$  is a complete classifying invariant for  $\text{Pic}_{\mathfrak{Q}}(X, \tau)$ . Moreover, this definition is in accordance with the result in Corollary 2.12 (2).

The structure of the “Quaternionic” line bundles can be investigated in a more direct way under some additional assumptions. Consider a representative  $\mathcal{L}$  in  $\text{Pic}_{\mathfrak{Q}}(X, \tau)$  such that the underlying line bundle is trivial in the complex category, i. e.  $\mathcal{L} \simeq X \times \mathbb{C}$ . In this situation a “Quaternionic” structure  $\Theta$  is fixed by a continuous map  $q : X \rightarrow \mathbb{U}(1)$  such that  $\Theta : (x, \lambda) \mapsto (\tau(x), q(x)\bar{\lambda})$ . The “Quaternionic” constraint requires  $q(\tau(x))\overline{q(x)} = -1$ , or equivalently  $q(\tau(x)) = -q(x)$ , for all  $x \in X$ . Evidently, this condition cannot be verified if the involution  $\tau$  is not free. The identification  $\mathbb{U}(1) \simeq \mathbb{S}^1$  says that the “Quaternionic” structure  $\Theta$  is specified by a  $\mathbb{Z}_2$ -equivariant map  $q$  from the involutive space  $(X, \tau)$  into the one-dimensional sphere endowed with the antipodal free involution  $\mathbb{S}^{0,2}$ . Since equivariant homotopy deformations produce isomorphic “Quaternionic” vector bundles we can conclude that each equivariant homotopy class in  $[X, \mathbb{S}^{0,2}]_{\mathbb{Z}_2}$  identifies a “Quaternionic” structure on the product line bundle  $\mathcal{L} \simeq X \times \mathbb{C}$ . Finally, we notice that the “Quaternionic” structures induced by an equivariant map  $q$  and its opposite  $-q$  are  $\mathfrak{Q}$ -isomorphic. In conclusion we obtained that “Quaternionic” structures on a product line bundle are classified by  $[X, \mathbb{S}^{0,2}]_{\mathbb{Z}_2}/\mathbb{Z}_2$  where the  $\mathbb{Z}_2$ -action is induced by the multiplication by  $-1$ . We are now in position to state the next result.

**Proposition 3.4.** *Let  $(X, \tau)$  be an involutive space which verifies the following condition:*

- (a)  $X^\tau = \emptyset$ ;
- (b)  $H^2(X, \mathbb{Z})$  has no torsion;
- (c) *The pullback homomorphism  $\tau^* : H^2(X, \mathbb{Z}) \rightarrow H^2(X, \mathbb{Z})$  acts as the identity.*

*Then the following bijection*

$$\text{Pic}_{\mathfrak{Q}}(X, \tau) \simeq [X, \mathbb{S}^{0,2}]_{\mathbb{Z}_2}/\mathbb{Z}_2$$

*holds true. The result is evidently valid under the stronger condition  $H^2(X, \mathbb{Z}) = 0$ , which implies (b) and (c).*

*Proof.* Condition (a) is necessary for the existence of “Quaternionic” line bundles. Let  $\mathcal{L}$  be an element in  $\text{Pic}_{\mathfrak{Q}}(X, \tau)$ . Conditions (b) and (c) are sufficient to ensure that  $\mathcal{L} \simeq X \times \mathbb{C}$  in the complex category. In fact the presence of a “Quaternionic” structure induces an isomorphism of complex vector bundles

$$\tau^* \mathcal{L} \simeq \overline{\mathcal{L}}$$

where  $\overline{\mathcal{L}}$  is the *conjugated* complex line bundle obtained by reversing the complex structure in the fibers of  $\mathcal{L}$ . This isomorphism leads to the constraint

$$\tau^* c_1(\mathcal{L}) = -c_1(\mathcal{L}) \tag{3.1}$$

for the first Chern class of  $\mathcal{L}$ . Condition (c) implies  $2c_1(\mathcal{L}) = 0$  and condition (b) assures that  $c_1(\mathcal{L}) = 0$ . This fact forces  $\mathcal{L}$  to be trivial (in the complex category) since complex line bundles are completely classified by the first Chern class. ■

One interesting aspect of the theory of “Quaternionic” line bundles is that there is no natural candidate for the definition of a *trivial* element in  $\text{Pic}_{\mathfrak{Q}}(X, \tau)$ !

**3.2. Classification over involutive spheres.** We discuss in this section the classification of “Quaternionic” line bundles over the involutive spheres of type  $\mathbb{S}^{0,d}$ .

**Example 3.5** (The Dupont  $\mathfrak{Q}$ -line bundle over  $\mathbb{S}^{0,1}$ ). The zero-dimensional sphere  $\mathbb{S}^{0,1}$  coincides with the *two-points* space  $\{-1, +1\}$  endowed with the flip-action  $\theta_{0,1} : \pm 1 \mapsto \mp 1$ . We can construct a

“Quaternionic” line bundle over  $\mathbb{S}^{0,1}$  as follows: Let  $\mathcal{L} := \{-1, +1\} \times \mathbb{C}$  be the product line bundle and define  $\Theta_0 : \mathcal{L} \rightarrow \mathcal{L}$  by

$$\Theta_0 : (\pm 1, \lambda) \mapsto (\mp 1, \pm \bar{\lambda}), \quad \lambda \in \mathbb{C}.$$

This example has been introduced for the first time by J. L. Dupont in [Du]. In particular it proves that,  $\text{Pic}_{\mathfrak{Q}}(\mathbb{S}^{0,1}) \neq \emptyset$ . The next natural question is whether there are other elements in  $\text{Pic}_{\mathfrak{Q}}(\mathbb{S}^{0,1})$  distinguished from the Dupont’s example. A way to answer this question is to use the characterization of Corollary 3.2. To compute the group  $H_{\mathbb{Z}_2}^2(\mathbb{S}^{0,1}, \mathbb{Z}(1))$  let us start with the ordinary cohomology

$$H^k(\{-1, +1\}, \mathbb{Z}) \simeq \begin{cases} \mathbb{Z}^2 & \text{if } k = 0 \\ 0 & \text{if } k \neq 0 \end{cases}$$

and with the equivariant cohomology with fixed coefficients

$$H_{\mathbb{Z}_2}^k(\mathbb{S}^{0,1}, \mathbb{Z}) \simeq H^k(\{*\}, \mathbb{Z}) \simeq \begin{cases} \mathbb{Z} & \text{if } k = 0 \\ 0 & \text{if } k \neq 0 \end{cases}$$

The application of the exact sequence in [Go, Proposition 2.3] produces  $H_{\mathbb{Z}_2}^2(\mathbb{S}^{0,1}, \mathbb{Z}) \simeq H_{\mathbb{Z}_2}^2(\mathbb{S}^{0,1}, \mathbb{Z}(1))$  and consequently

$$\text{Pic}_{\mathfrak{Q}}(\mathbb{S}^{0,1}) \simeq H_{\mathbb{Z}_2}^2(\mathbb{S}^{0,1}, \mathbb{Z}(1)) = 0$$

contains a single element that can be identified with the Dupont’s example. The unicity (up to isomorphism) of this representative can be checked also in a more direct way. First of all we observe that every complex vector bundle over  $\{-1, +1\}$  is necessarily trivial. This means that every “Quaternionic” line bundle over  $\mathbb{S}^{0,1}$  is builded over the underlying complex line bundle  $\mathcal{L} := \{-1, +1\} \times \mathbb{C}$ . Any “Quaternionic” structure on  $\mathcal{L}$  can be specified by a  $t \in [0, 2\pi)$  according to

$$\Theta_t : (\pm 1, \lambda) \mapsto (\mp 1, \pm e^{it} \bar{\lambda}), \quad \lambda \in \mathbb{C}.$$

On the other hand a  $\mathfrak{Q}$ -isomorphism between two “Quaternionic” structures  $\Theta_t$  and  $\Theta_{t'}$  is provided by a map

$$f_\phi : (\pm 1, \lambda) \mapsto (\pm 1, \phi(\pm 1)\lambda), \quad \lambda \in \mathbb{C}$$

subjected to the equivariant condition  $f_\phi \circ \Theta_t = \Theta_{t'} \circ f_\phi$ . For each pair  $t, t' \in [0, 2\pi)$  such a map is fixed by the prescription  $\phi(+1) := +1$  and  $\phi(-1) := e^{i(t'-t)}$ . This proves directly that each “Quaternionic” line bundle over  $\{-1, +1\}$  is isomorphic to the Dupont’s example which corresponds to  $t = 0$ .  $\blacktriangleleft$

**Example 3.6** ( $\mathfrak{Q}$ -line bundles over  $\mathbb{S}^{0,d+1}$  with  $d \geq 3$ ). Recall that  $\mathbb{S}^{0,d+1}$  agrees with  $\mathbb{S}^d$  as topological space and  $H^2(\mathbb{S}^d, \mathbb{Z}) = 0$  for  $d \geq 3$ . Then, we can use the result in Proposition 3.4 to state that

$$\text{Pic}_{\mathfrak{Q}}(\mathbb{S}^{0,d+1}) \simeq [\mathbb{S}^{0,d+1}, \mathbb{S}^{0,2}]_{\mathbb{Z}_2} / \mathbb{Z}_2, \quad d \geq 3.$$

However, an analysis of the equivariant homotopy shows that

$$[\mathbb{S}^{0,d+1}, \mathbb{S}^{0,2}]_{\mathbb{Z}_2} = \emptyset, \quad d \geq 2. \quad (3.2)$$

and this leads to the conclusion that it is not possible to build “Quaternionic” line bundle over involutive sphere  $\mathbb{S}^{0,d+1}$  of dimension bigger than two. As a matter of completeness, let us justify equation (3.2). The involutive space  $\mathbb{S}^{0,d+1}$  can be seen as the total space of a  $\mathbb{Z}_2$ -sphere bundle  $\mathbb{S}^{0,d+1} \rightarrow \mathbb{R}P^d$  and it is well known that the Stiefel-Whitney class of this bundle  $w_1(\mathbb{S}^{0,d+1})$  is given by the non-trivial element of  $H^1(\mathbb{R}P^d, \mathbb{Z}_2) \simeq \mathbb{Z}_2$  for all  $d \geq 1$ . Let us assume the existence of a  $\mathbb{Z}_2$ -equivariant map  $q : \mathbb{S}^{0,d+1} \rightarrow \mathbb{S}^{0,2}$ . Such a map would induce a map  $\tilde{q} : \mathbb{R}P^d \rightarrow \mathbb{R}P^1$  on the orbit spaces and an isomorphism  $\mathbb{S}^{0,d+1} \simeq \tilde{q}^* \mathbb{S}^{0,2}$  of bundles over  $\mathbb{R}P^d$  where  $\tilde{q}^* \mathbb{S}^{0,2}$  is the pullback of the  $\mathbb{Z}_2$ -sphere bundles  $\mathbb{S}^{0,2} \rightarrow \mathbb{R}P^1$ . By functoriality, this would imply  $w_1(\mathbb{S}^{0,d+1}) = w_1(\tilde{q}^* \mathbb{S}^{0,2}) = \tilde{q}^* w_1(\mathbb{S}^{0,2})$  showing that  $\tilde{q}^* w_1(\mathbb{S}^{0,2})$  is not trivial. On the other hand one has that

$$[\mathbb{R}P^d, \mathbb{R}P^1] \simeq [\mathbb{R}P^d, K(\mathbb{Z}, 1)] \simeq H^1(\mathbb{R}P^d, \mathbb{Z}) = 0, \quad d \geq 2$$



where  $\mathbb{R}P^1$  has been identified with the Eilenberg-MacLane space  $K(\mathbb{Z}, 1)$ . Therefore  $\tilde{q}$  must be homotopy equivalent to the constant map and this contradicts the non-triviality of  $\tilde{q}^*w_1(\mathbb{S}^{0,2})$ . This contradiction shows that the set of  $\mathbb{Z}_2$ -equivariant map from  $\mathbb{S}^{0,d+1}$  into  $\mathbb{S}^{0,2}$  must be empty in agreement with (3.2).  $\blacktriangleleft$

**Example 3.7** ( $\mathbb{Q}$ -line bundles over  $\mathbb{S}^{0,2}$ ). Let us start by proving that  $\text{Pic}_{\mathbb{Q}}(\mathbb{S}^{0,2}) \neq \emptyset$ . Since  $H^2(\mathbb{S}^1, \mathbb{Z}) = 0$  we can use the same argument as in Proposition 3.4. The vanishing of the first Chern class implies that each (possible) “Quaternionic” line bundle over  $\mathbb{S}^{0,2}$  must be build over the underlying product bundle  $\mathcal{L}_0 := \mathbb{S}^1 \times \mathbb{C}$ . Then, a (possible) “Quaternionic” structure  $\Theta_q : \mathcal{L}_0 \rightarrow \mathcal{L}_0$  must be specified by a map  $q : \mathbb{S}^1 \rightarrow \mathbb{U}(1)$  which fulfills the constraint  $q(-k) = -q(k)$  for all  $k \in \mathbb{S}^1$  according to the recipe

$$\Theta_q : (k, \lambda) \mapsto (-k, q(k)\bar{\lambda}), \quad (k, \lambda) \in \mathbb{S}^1 \times \mathbb{C}.$$

A standard choice for such a map is the following:

$$\begin{aligned} q_0 : \mathbb{S}^1 &\longrightarrow \mathbb{U}(1) \\ (k_0, k_1) &\longmapsto k_0 + i k_1. \end{aligned} \quad (3.3)$$

The existence of  $q_0$  assures that  $\text{Pic}_{\mathbb{Q}}(\mathbb{S}^{0,2})$  is nonempty and so the classification of Corollary 3.2 applies. The computation in Proposition B.2 gives

$$\text{Pic}_{\mathbb{Q}}(\mathbb{S}^{0,2}) = 0,$$

namely there is only one isomorphism class of “Quaternionic” line bundles over  $\mathbb{S}^{0,2}$  with a representative given by  $(\mathcal{L}_0, q_0)$ . It is possible to verify this claim in a more direct way. A  $\mathbb{Q}$ -isomorphism between two “Quaternionic” structures  $\Theta_q$  and  $\Theta_{q_0}$  is specified by a map  $\phi : \mathbb{S}^1 \rightarrow \mathbb{U}(1)$  such that

$$f_\phi : (k, \lambda) \mapsto (k, \phi(k)\lambda), \quad (k, \lambda) \in \mathbb{S}^1 \times \mathbb{C},$$

and subjected to the equivariance condition  $f_\phi \circ \Theta_q = \Theta_{q_0} \circ f_\phi$ . This translates into

$$q_0(k) = \phi(-k) \phi(k) q(k) \quad k \in \mathbb{S}^1.$$

In a general way the map  $q$  can be related to  $q_0$  by a relation of the form

$$q(k) = q_0(k)^n e^{i 2\pi \chi(k)} \quad k \in \mathbb{S}^1 \quad (3.4)$$

for some  $n \in \mathbb{Z}$  and  $\chi : \mathbb{S}^1 \rightarrow \mathbb{R}$ . The “Quaternionic” constraints  $-1 = q_0(-k)\overline{q_0(k)}$  and  $-1 = q(-k)\overline{q(k)}$  impose the condition

$$e^{i 2\pi [\chi(-k) - \chi(k)]} = (-1)^{n+1} \quad k \in \mathbb{S}^1.$$

If  $n$  were even  $\Delta\chi(k) := \chi(-k) - \chi(k)$  should take values in  $\mathbb{Z} + \frac{1}{2}$ . However,  $\Delta\chi$  is continuous and so  $\Delta\chi(k)$  should take a constant value  $N + \frac{1}{2}$  for some  $N \in \mathbb{Z}$  and for all  $k$ . But this is incompatible with the parity of  $\Delta\chi$  which requires  $\Delta\chi(-k) = -\Delta\chi(k)$ . Then  $n$  must be odd and  $\Delta\chi$  must take values in  $\mathbb{Z}$ . Again, the continuity and the parity of  $\Delta\chi$  imply that  $\Delta\chi(k) = 0$ , or equivalently  $\chi(-k) = \chi(k)$  for all  $k$ . This allows to rewrite equation (3.4) in the form

$$q_0(k) = q_0(k)^{-2n} e^{-i 2\pi \chi(k)} q(k) = \phi(-k) \phi(k) q(k) \quad k \in \mathbb{S}^1$$

with  $\phi(k) := (-1)^{\frac{n}{2}} q_0(k)^{-n} e^{-i \pi \chi(k)}$ . The last equation shows that any “Quaternionic” structure  $\Theta_q$  is  $\mathbb{Q}$ -isomorphic to the standard “Quaternionic” structure  $\Theta_{q_0}$ .  $\blacktriangleleft$

**Example 3.8** ( $\mathbb{Q}$ -line bundles over  $\mathbb{S}^{0,3}$ ). Again, let us start by showing that  $\text{Pic}_{\mathbb{Q}}(\mathbb{S}^{0,3}) \neq \emptyset$ . This case is richer than those described in Example 3.6 and Example 3.7 since  $H^2(\mathbb{S}^2, \mathbb{Z}) \simeq \mathbb{Z}$  which in turn implies that for each  $\ell \in \mathbb{Z}$  there is an inequivalent complex line bundle  $\mathcal{L}_\ell \rightarrow \mathbb{S}^2$ , specified by the Chern class  $\ell$ . In principle each of these  $\mathcal{L}_\ell$  can be used as the underlying line bundle for a “Quaternionic” structure. Equation (3.2) says that it is not possible to endow the trivial line bundle

$\mathcal{L}_0 := \mathbb{S}^2 \times \mathbb{C}$  with a “Quaternionic” structure. Let us investigate the case  $\ell = 1$ . A representative for  $\mathcal{L}_1 \rightarrow \mathbb{S}^2$  can be constructed from the family of *Hopf projections*

$$P_{\text{Hopf}}(k_0, k_1, k_2) := \frac{1}{2} \left( \mathbb{1}_2 + \sum_{j=0}^2 k_j \sigma_{j+1} \right), \quad k := (k_0, k_1, k_2) \in \mathbb{S}^2 \quad (3.5)$$

according to

$$\mathcal{L}_1 := \bigsqcup_{k \in \mathbb{S}^2} \text{Ran}(P_{\text{Hopf}}(k))$$

where  $\sigma_1, \sigma_2$  and  $\sigma_3$  are the Pauli matrices and  $\text{Ran}(P_{\text{Hopf}}(k)) \subset \mathbb{C}^2$  is the one-dimensional subspace spanned by  $P_{\text{Hopf}}(k)$ . Moreover,  $\mathcal{L}_1$  has a “Quaternionic” structure induced by the anti-linear symmetry

$$\Theta_1 P_{\text{Hopf}}(k_0, k_1, k_2) \Theta_1^* = P_{\text{Hopf}}(-k_0, -k_1, -k_2), \quad \Theta_1 := \sigma_2 \circ C$$

where  $C$  is the operator which implements the complex conjugation on  $\mathbb{C}^2$  (cf. Section 5). The existence of the “Quaternionic” line bundle  $(\mathcal{L}_1, \Theta_1)$  assures that  $\text{Pic}_{\mathbb{Q}}(\mathbb{S}^{0,3})$  is non-empty and so the classification of Corollary 3.2 applies. Proposition B.2 provides

$$\text{Pic}_{\mathbb{Q}}(\mathbb{S}^{0,3}) \simeq H^2(\mathbb{S}^{0,3}, \mathbb{Z}(1)) \simeq \mathbb{Z},$$

namely there are  $\mathbb{Z}$  distinct isomorphism classes of  $\mathbb{Q}$ -line bundles over  $\mathbb{S}^{0,3}$ . A look to the exact sequence [Go, Proposition 2.3.]

$$\begin{array}{ccccccc} H_{\mathbb{Z}_2}^1(\mathbb{S}^{0,3}, \mathbb{Z}) & \longrightarrow & H_{\mathbb{Z}_2}^2(\mathbb{S}^{0,3}, \mathbb{Z}(1)) & \xrightarrow{f} & H^2(\mathbb{S}^2, \mathbb{Z}) & \longrightarrow & H_{\mathbb{Z}_2}^2(\mathbb{S}^{0,3}, \mathbb{Z}) \longrightarrow H_{\mathbb{Z}_2}^3(\mathbb{S}^{0,3}, \mathbb{Z}(1)) \\ \parallel & & \parallel & & \parallel & & \parallel \\ 0 & & \mathbb{Z} & & \mathbb{Z} & & \mathbb{Z}_2 \end{array}$$

shows that the map  $f$  which forgets the  $\mathbb{Z}_2$ -structure acts as the multiplication by 2. By combining this fact with the isomorphism  $c_1 : \text{Pic}_{\mathbb{C}}(\mathbb{S}^2) \simeq H^2(\mathbb{S}^2, \mathbb{Z})$  induced by the first Chern class and the isomorphism  $c_1^{\mathbb{R}} : \text{Pic}_{\mathbb{R}}(\mathbb{S}^{0,3}) \rightarrow H_{\mathbb{Z}_2}^2(\mathbb{S}^{0,3}, \mathbb{Z}(1))$  induced by the first “Real” Chern class one concludes that there is an *isomorphisms of groups*

$$\begin{array}{ccc} \text{Pic}_{\mathbb{R}}(\mathbb{S}^{0,3}) & \xrightarrow{\simeq} & 2\mathbb{Z} \\ [\mathcal{L}_{\mathbb{R}}] & \mapsto & c_1(\mathcal{L}_{\mathbb{R}}) \end{array}$$

where  $\mathcal{L}_{\mathbb{R}}$  is any “Real” line bundle over  $\mathbb{S}^{0,3}$ . The characterization of Theorem 3.1 which describes  $\text{Pic}_{\mathbb{Q}}(\mathbb{S}^{0,3})$  as a torsor over  $\text{Pic}_{\mathbb{R}}(\mathbb{S}^{0,3})$  implies the *bijection* of sets given by

$$\begin{array}{ccc} \text{Pic}_{\mathbb{Q}}(\mathbb{S}^{0,3}) & \xrightarrow{\simeq} & 2\mathbb{Z} + 1 \\ [\mathcal{L}_{\mathbb{Q}}] \simeq [\mathcal{L}_{\mathbb{R}} \otimes \mathcal{L}_1] & \mapsto & c_1(\mathcal{L}_{\mathbb{Q}}) = c_1(\mathcal{L}_{\mathbb{R}}) + c_1(\mathcal{L}_1). \end{array}$$

We point out that the “Quaternionic” line bundle  $\mathcal{L}_1$  has been used as the *reference* element for the identification of  $\text{Pic}_{\mathbb{Q}}(\mathbb{S}^{0,3})$  as a  $\text{Pic}_{\mathbb{R}}(\mathbb{S}^{0,3})$ -orbit. Summarizing, one has:

- (1) *A complex line bundle can support a unique (up to isomorphisms) “Real” structure over the two-dimensional sphere endowed with the free antipodal action if and only if its first Chern class is even and different Chern classes distinguish between inequivalent “Real” line bundles;*
- (2) *A complex line bundle can support a unique (up to isomorphisms) “Quaternionic” structure over the two-dimensional sphere endowed with the free antipodal action if and only if its first Chern class is odd and different Chern classes distinguish between inequivalent “Quaternionic” line bundles. Moreover, for each  $\mathcal{L}_{\mathbb{Q}} \in \text{Pic}_{\mathbb{Q}}(\mathbb{S}^{0,3})$  is specified by its FKMM-invariant given by*

$$\kappa(\mathcal{L}_{\mathbb{Q}}) = \frac{c_1(\mathcal{L}_{\mathbb{Q}}) - 1}{2}.$$

*constructed according to Definition 3.3.*

Let us mention that that results similar to (1) and (2) have been recently derived in [GR] with a different technique based on the analysis of obstructions to the construction of trivializing frames. It is interesting to note that our derivation of (1) and (2) turns out to be no more than an exercise once Theorem 3.1 has been established. ◀

#### 4. TOPOLOGICAL CLASSIFICATION OF “QUATERNIONIC” VECTOR BUNDLES

In this section we analyze the role of the FKMM-invariant in the classification of the “Quaternionic” vector bundles. It turns out that the  $\kappa$ -invariant is an extremely efficient tool to solve the classification problem in *low* dimensions. Just to fix the terminology (for the present work) we say that:

**Definition 4.1** (Low dimension). *A topological space  $X$  which verifies Assumption 2.1 is said to be low dimensional if  $0 \leq d \leq 3$ .*

**4.1. Stable rank condition.** The stable rank condition for vector bundles expresses the pretty general fact that the non trivial topology can be concentrated in a sub-vector bundle of *minimal* rank. This minimal value depends on the dimensionality of the base space and on the category of vector bundles under consideration. For complex (as well as real or quaternionic) vector bundles the stable rank condition is a well-known result (see e. g. [Hus, Chapter 9, Theorem 1.2]). The proof of this result is based on an “obstruction-type argument” which provides the explicit construction of a certain *maximal* number of global sections [Hus, Chapter 2, Theorem 7.1]. The latter argument can be generalized to vector bundles over spaces with involution by means of the notion of  $\mathbb{Z}_2$ -CW-complex [Mat, AP] (see also [DG1, Section 4.5]). A  $\mathbb{Z}_2$ -CW-complex is a CW-complex made by cells of various dimension that carry a  $\mathbb{Z}_2$ -action. These  $\mathbb{Z}_2$ -cells can be only of two types: They are *fixed* if the action of  $\mathbb{Z}_2$  is trivial or they are *free* if they have no fixed points. Since this construction is modelled after the usual definition of CW-complex, just by replacing the “point” by “ $\mathbb{Z}_2$ -point”, (almost) all topological and homological properties valid for CW-complexes have their “natural” counterparts in the equivariant setting. The use of this technique is essential for the determination of the stable rank condition in the case of “Real” vector bundles [DG1, Theorem 4.25] and even rank “Quaternionic” vector bundles [DG2, Theorem 2.5]. In this section we discuss the generalization of the stable rank condition in the “Quaternionic” category in situation not covered by in [DG2, Theorem 2.5] (e. g.  $X^\tau$  of any codimension) and for odd rank vector bundles. We start with the even dimensional case.

**Theorem 4.2** (Stable condition: even rank). *Let  $(X, \tau)$  be an involutive space such that  $X$  has a finite  $\mathbb{Z}_2$ -CW-complex decomposition of dimension  $d$ . Each rank  $2m$  “Quaternionic” vector bundle  $(\mathcal{E}, \Theta)$  over  $(X, \tau)$  such that  $d \leq 4m - 3$  splits as*

$$\mathcal{E} \simeq \mathcal{E}_0 \oplus (X \times \mathbb{C}^{2(m-\sigma)}) \quad (4.1)$$

where  $\sigma := [\frac{d+2}{4}]$  (here  $[x]$  denotes the integer part of  $x \in \mathbb{R}$ ),  $\mathcal{E}_0$  is a (possible) non-trivial “Quaternionic” vector bundle of rank  $2\sigma$  and the remainder is the trivial “Quaternionic” vector bundle of rank  $2(m - \sigma)$ . As a consequence, one has

$$\text{Vec}_{\mathbb{Q}}^{2m}(X, \tau) \simeq \text{Vec}_{\mathbb{Q}}^{2\sigma}(X, \tau) \quad \forall m \geq \frac{d+3}{4} \quad (4.2)$$

and more specifically

$$\text{Vec}_{\mathbb{Q}}^{2m}(X, \tau) = 0 \quad \text{if } d = 0, 1 \quad \forall m \in \mathbb{N} \quad (4.3)$$

$$\text{Vec}_{\mathbb{Q}}^{2m}(X, \tau) \simeq \text{Vec}_{\mathbb{Q}}^{2\sigma}(X, \tau) \quad \text{if } 2 \leq d \leq 5 \quad \forall m \in \mathbb{N}. \quad (4.4)$$

*Proof.* The claim above generalizes that of [DG2, Theorem 2.5] because it includes also the cases: (a)  $X^\tau = \emptyset$  and, (b)  $X^\tau$  a  $\mathbb{Z}_2$ -CW-complex of dimension bigger than zero. These extensions are subject to the possibility of generalizing in the same direction the key construction described in the proof of [DG2, Proposition 2.7]. For the case (a) this generalization is trivial since we need only to assume that the number of fixed points is zero. At the first step of the inductive argument one must deal only with pairs of conjugated points  $\{x_j, \tau(x_j)\}$  and one can construct  $\mathbb{Q}$ -pairs of independent sections

$s_1, s_2$  over  $\{x_j, \tau(x_j)\}$  just by choosing  $s_1(x_j) = v_1$ ,  $s_1(\tau x_j) = v_2$ ,  $s_2(x_j) = \Theta(v_2)$ ,  $s_2(\tau x_j) = \Theta(v_1)$  with  $v_1, v_2 \in \mathbb{C}^{2m} \setminus \{0\}$  and  $v_2 \neq \lambda \Theta(v_1)$  for some  $\lambda \in \mathbb{C}$ . At this point the rest of the argument follows exactly as in the proof of [DG2, Proposition 2.7]. The validity of the case (b) has been already anticipated and justified in [DG2, Remark 2.4].  $\blacksquare$

**Remark 4.3** (Stable rank condition for the “Real” case). The same argument used in the proof of Theorem 4.2 can be applied to extend the stable rank condition for “Real” vector bundles [DG1, Theorem 4.25] to the case of a free action. In summary, under the conditions: (a)  $X^\tau = \emptyset$  or (b’)  $X^\tau$  a  $\mathbb{Z}_2$ -CW-complex of dimension zero, one has that

$$\mathrm{Vec}_{\Re}^m(X, \tau) \simeq \mathrm{Vec}_{\Re}^\sigma(X, \tau) \quad \forall m \geq \frac{d+1}{2} \quad (4.5)$$

where  $\sigma := \lceil \frac{d}{2} \rceil$ , and more specifically

$$\mathrm{Vec}_{\Re}^m(X, \tau) = 0 \quad \text{if } d = 0, 1 \quad \forall m \in \mathbb{N} \quad (4.6)$$

$$\mathrm{Vec}_{\Re}^m(X, \tau) \simeq \mathrm{Vec}_{\Re}^1(X, \tau) \quad \text{if } 2 \leq d \leq 3 \quad \forall m \in \mathbb{N}. \quad (4.7)$$

The situation turns out to be quite different when  $X^\tau$  is a  $\mathbb{Z}_2$ -CW-complex of dimension bigger than zero as pointed out in [DG1, Remark 4.24].  $\blacktriangleleft$

The odd rank case requires a slight different analysis and strongly depends on the existence of a “Quaternionic” line bundle. The following result provides the key argument.

**Lemma 4.4.** *Let  $(X, \tau)$  be an involutive space such that  $X^\tau = \emptyset$  and  $\mathrm{Pic}_{\mathbb{Q}}(X, \tau) \neq \emptyset$ . Then, one has a bijection*

$$\mathrm{Vec}_{\mathbb{Q}}^m(X, \tau) \simeq \mathrm{Vec}_{\Re}^m(X, \tau) \quad \forall m \in \mathbb{N}.$$

Moreover,

$$\mathrm{Vec}_{\mathbb{Q}}^{2m+1}(X, \tau) = \emptyset \quad \Leftrightarrow \quad \mathrm{Pic}_{\mathbb{Q}}(X, \tau) = \emptyset.$$

*Proof.* The proof of the second claim is easy. If  $[\mathcal{L}_{\mathbb{Q}}] \in \mathrm{Pic}_{\mathbb{Q}}(X, \tau) \neq \emptyset$  one has that  $\mathcal{L}_{\mathbb{Q}} \oplus (X \times \mathbb{C}^{2m})$  is a “Quaternionic” vector bundle of rank  $2m+1$  and so  $\mathrm{Vec}_{\mathbb{Q}}^{2m+1}(X, \tau) \neq \emptyset$ . Conversely, if  $[(\mathcal{E}, \Theta)] \in \mathrm{Vec}_{\mathbb{Q}}^{2m+1}(X, \tau) \neq \emptyset$ , then the line bundle  $\det(\mathcal{E})$  inherits a “Quaternionic” structure given by  $\det(\Theta)$  and so  $\mathrm{Pic}_{\mathbb{Q}}(X, \tau) \neq \emptyset$ . Let now assume that  $\mathrm{Pic}_{\mathbb{Q}}(X, \tau) \neq \emptyset$  and choose a representative  $\mathcal{L}_{\mathbb{Q}}$  of some element in  $\mathrm{Pic}_{\mathbb{Q}}(X, \tau)$ . The map

$$\begin{aligned} \mathcal{L}_{\mathbb{Q}} \otimes : \mathrm{Vec}_{\mathbb{Q}}^m(X, \tau) &\xrightarrow{\simeq} \mathrm{Vec}_{\Re}^m(X, \tau) \\ [\mathcal{E}_{\mathbb{Q}}] &\longmapsto [\mathcal{L}_{\mathbb{Q}} \otimes \mathcal{E}_{\mathbb{Q}}] \end{aligned}$$

turns out to be a bijection of sets for each  $m \in \mathbb{N}$ . This fact can be proved with an argument similar to that in the proof of Theorem 3.1 and is based on the observation that  $\mathcal{L}_{\mathbb{Q}} \otimes \mathcal{L}_{\mathbb{Q}}^*$  is equivalent to the trivial element in  $\mathrm{Pic}_{\Re}(X, \tau)$ .  $\blacksquare$

Let us point out that the condition  $\mathrm{Vec}_{\mathbb{Q}}^{2m}(X, \tau) \neq \emptyset$  is always guaranteed by the existence of the trivial “Quaternionic” product bundle  $X \times \mathbb{C}^{2m} \rightarrow X$ . The next result follows by combining Lemma 4.4 and the stable rank condition for “Real” vector bundles discussed in Remark 4.3.

**Theorem 4.5** (Stable condition: odd rank). *Let  $(X, \tau)$  be an involutive space such that  $X$  has a finite  $\mathbb{Z}_2$ -CW-complex decomposition of dimension  $d$  and  $X^\tau = \emptyset$ . Assume also  $\mathrm{Pic}_{\mathbb{Q}}(X, \tau) \neq \emptyset$ . Then*

$$\mathrm{Vec}_{\mathbb{Q}}^{2m+1}(X, \tau) \simeq \mathrm{Vec}_{\mathbb{Q}}^\sigma(X, \tau) \quad \forall m \geq \frac{d-1}{4}$$

where  $\sigma := \lceil \frac{d}{2} \rceil$ . In low dimensions this provides

$$\mathrm{Vec}_{\mathbb{Q}}^{2m-1}(X, \tau) = 0 \quad \text{if } d = 0, 1 \quad \forall m \in \mathbb{N} \quad (4.8)$$

$$\mathrm{Vec}_{\mathbb{Q}}^{2m-1}(X, \tau) = \mathrm{Pic}_{\mathbb{Q}}(X, \tau) \quad \text{if } d = 2, 3 \quad \forall m \in \mathbb{N} \quad (4.9)$$

$$\mathrm{Vec}_{\mathbb{Q}}^{2m-1}(X, \tau) \simeq \mathrm{Vec}_{\mathbb{Q}}^2(X, \tau) \quad \text{if } d = 4, 5 \quad \forall m \in \mathbb{N}. \quad (4.10)$$

We point out that the 0 in the (4.8) refers to the existence of a unique element which could also be different from a (trivial) product vector bundle.

**4.2. Injectivity in low dimension.** The FKMM-invariant introduced in Definition 2.8 provides a map

$$\kappa : \text{Vec}_{\mathbb{Q}}^{2m}(X, \tau) \longrightarrow H_{\mathbb{Z}_2}^2(X|X^\tau, \mathbb{Z}(1))$$

which can be used to classify “Quaternionic” vector bundles over  $(X, \tau)$ . A remarkable property is that, under certain assumptions on  $(X, \tau)$ ,  $\kappa$  provides a *natural injection*. A similar result has been already proved in [DG2, Theorem 1.1] under the hypothesis that:  $(\alpha)$   $(X, \tau)$  is a low dimensional space according to Definition 4.1 ; and  $(\beta)$   $(X, \tau)$  is an FKMM-space [DG2, Definition 1.1]. While  $(\alpha)$  is indispensable (when  $d$  grows more invariants are needed to distinguish between inequivalent “Quaternionic” vector bundles, cf. [DG2, Theorem 1.3]), the assumption  $(\beta)$  can be weakened. First of all one can renounce to the requirement  $H_{\mathbb{Z}_2}^2(X, \mathbb{Z}(1)) = 0$  proper of an FKMM-space. Second, one can consider (separately) the case  $X^\tau \neq \emptyset$  of any codimension and the case  $X^\tau = \emptyset$ . The first step for the study of the injectivity of  $\kappa$  consists in the following generalization (and simplification) of [DG2, Lemma 4.1].

**Lemma 4.6.** *Let  $(X, \tau)$  be a space with involution such that  $X^\tau \neq \emptyset$ . Let  $(\mathcal{E}_1, \Theta_1)$  and  $(\mathcal{E}_2, \Theta_2)$  be rank 2 “Quaternionic” vector bundles on  $(X, \tau)$  such that:*

- (a) *There is an isomorphism  $\Psi : \mathcal{E}_1|_{X^\tau} \rightarrow \mathcal{E}_2|_{X^\tau}$  of “Quaternionic” vector bundles over  $X^\tau$ ;*
- (b)  *$\kappa(\mathcal{E}_1, \Theta_1) = \kappa(\mathcal{E}_2, \Theta_2)$  in  $H_{\mathbb{Z}_2}^2(X|X^\tau, \mathbb{Z}(1))$ .*

*Then, there is an isomorphism of “Real” line bundles  $\psi : (\det(\mathcal{E}_1), \det(\Theta_1)) \rightarrow (\det(\mathcal{E}_2), \det(\Theta_2))$  on  $(X, \tau)$  such that  $\det(\Psi) = \psi|_{X^\tau}$ .*

*Proof.* Let  $j = 1, 2$ . Recall that the two determinant “Real” line bundles  $(\det(\mathcal{E}_j), \det(\Theta_j))$  associated to  $(\mathcal{E}_j, \Theta_j)$  admit unique *canonical*  $\mathfrak{R}$ -sections  $s_{\mathcal{E}_j} : X^\tau \rightarrow \mathbb{S}(\det(\mathcal{E}_j)|_{X^\tau})$  for their restrictions on  $X^\tau$  (Proposition 2.5). Assumption (a) and the uniqueness of the canonical section assure that the  $\mathfrak{R}$ -isomorphism  $\det(\Psi) : \det(\mathcal{E}_1)|_{X^\tau} \rightarrow \det(\mathcal{E}_2)|_{X^\tau}$  intertwines the canonical sections in the sense that

$$s_{\mathcal{E}_2} = \det(\Psi) \circ s_{\mathcal{E}_1}.$$

According to Definition 2.8, the invariants  $\kappa(\mathcal{E}_j, \Theta_j)$  agree with the isomorphism classes of the pairs  $(\det(\mathcal{E}_j), s_{\mathcal{E}_j})$ . Thus, by assumption (b), there is also an  $\mathfrak{R}$ -isomorphism  $\psi : \det(\mathcal{E}_1) \rightarrow \det(\mathcal{E}_2)$  such that

$$s_{\mathcal{E}_2} = \psi|_{X^\tau} \circ s_{\mathcal{E}_1}.$$

The difference between the isomorphisms  $\det(\Psi)$  and  $\psi|_{X^\tau}$  is uniquely specified by a map  $u : X^\tau \rightarrow \mathbb{U}(1)$  according to the relation  $\det(\Psi) = \psi|_{X^\tau} \cdot u$ . This implies

$$s_{\mathcal{E}_2} = \det(\Psi) \circ s_{\mathcal{E}_1} = (\psi|_{X^\tau} \cdot u) \circ s_{\mathcal{E}_1} = (\psi|_{X^\tau} \circ s_{\mathcal{E}_1}) \cdot u = s_{\mathcal{E}_2} \cdot u$$

namely  $u$  is the *constant map*  $u \equiv 1$  on  $X^\tau$  and so  $\det(\Psi) = \psi|_{X^\tau}$ . ■

Let us consider now a pair of rank 2 “Quaternionic” vector bundles  $(\mathcal{E}_1, \Theta_1)$  and  $(\mathcal{E}_2, \Theta_2)$  over  $(X, \tau)$  related by an  $\mathfrak{R}$ -isomorphism  $\psi : (\det(\mathcal{E}_1), \det(\Theta_1)) \rightarrow (\det(\mathcal{E}_2), \det(\Theta_2))$  of the respective determinant line bundles. The construction in [DG2, Lemma 4.1] allows us to define the locally trivial fiber bundle

$$\mathcal{SU}(\mathcal{E}_1, \mathcal{E}_2, \psi) := \bigsqcup_{x \in X} \left\{ \Upsilon_x : \mathcal{E}_1|_x \rightarrow \mathcal{E}_2|_x \left| \begin{array}{l} \text{vector bundle isomorphism} \\ \text{such that } \det(\Upsilon_x) = \psi|_x \end{array} \right. \right\}$$

whose typical fiber is modeled out of  $\text{SU}(2)$ . It has been proved in [DG2, Lemma 4.1] that:

- (1) There is a natural involution on  $\mathcal{SU}(\mathcal{E}_1, \mathcal{E}_2, \psi)$  covering  $\tau$  which can be identified with the standard involution on  $\text{SU}(2)$  given by  $\mu : u \mapsto -Q\bar{u}Q$  (see [DG2, Remark 2.1] for more details) on the fixed point set  $X^\tau$ ;
- (2) The set of equivariant sections of  $\mathcal{SU}(\mathcal{E}_1, \mathcal{E}_2, \psi)$  is in bijection with the set of  $\mathfrak{Q}$ -isomorphisms  $\Upsilon : \mathcal{E}_1 \rightarrow \mathcal{E}_2$  such that  $\det(\Upsilon) = \psi$ .

We are now in position to prove the following crucial result.



**Theorem 4.7** (Injectivity: even rank case). *Let  $(X, \tau)$  be a low dimensional involutive space in the sense of Definition 4.1. Then:*

(1) *If  $X^\tau \neq \emptyset$  the map*

$$\kappa : \text{Vec}_{\mathfrak{Q}}^{2m}(X, \tau) \longrightarrow H_{\mathbb{Z}_2}^2(X|X^\tau, \mathbb{Z}(1)), \quad m \in \mathbb{N}$$

*given by the FKMM-invariant provides a natural injective group homomorphism;*

(2) *If  $X^\tau = \emptyset$  the map*

$$\kappa : \text{Vec}_{\mathfrak{Q}}^{2m}(X, \tau) \longrightarrow H_{\mathbb{Z}_2}^2(X, \mathbb{Z}(1)), \quad m \in \mathbb{N}$$

*given by the identification of  $\kappa$  with the first “Real” Chern class of the associated determinant line bundle (cf. Corollary 2.12 (2)) provides a natural injective group homomorphism. In addition, this is an isomorphism if  $\text{Pic}_{\mathfrak{Q}}(X, \tau) \neq \emptyset$ .*

*Proof.* For  $d = 0, 1$  the claim is trivially true in both cases since  $\text{Vec}_{\mathfrak{Q}}^{2m}(X, \tau) = 0$  in view (4.3). For  $d = 2, 3$  the bijection  $\text{Vec}_{\mathfrak{Q}}^{2m}(X, \tau) \simeq \text{Vec}_{\mathfrak{Q}}^2(X, \tau)$  in equation (4.4) reduces the proof to the fact that  $\kappa$  is injective on  $\text{Vec}_{\mathfrak{Q}}^2(X, \tau)$ .

Let us start with the case (1). Let  $(\mathcal{E}_1, \Theta_1)$  be a “Quaternionic” vector bundle of rank 2 over  $(X, \tau)$ . The restriction  $\mathcal{E}_1|_{X^\tau}$  is isomorphic to a line bundle with quaternionic fiber over  $X^\tau$  [DG2, Proposition 2.2]. Since the dimension of  $X^\tau$  is less than four one has that  $\text{Vec}_{\mathbb{H}}^1(X^\tau) = 0$  as a consequence of the stable rank condition for  $\mathbb{H}$ -vector bundles [Hus, Chapter 9, Theorem 1.2]. Therefore  $\mathcal{E}_1|_{X^\tau}$  is  $\mathfrak{Q}$ -trivial. Let  $(\mathcal{E}_2, \Theta_2)$  be a second “Quaternionic” vector bundle of rank 2 over  $(X, \tau)$ . Since also  $\mathcal{E}_2|_{X^\tau}$  turns out to be  $\mathfrak{Q}$ -trivial for the same reason, there is an isomorphism of  $\mathfrak{Q}$ -bundles  $\Psi : \mathcal{E}_1|_{X^\tau} \rightarrow \mathcal{E}_2|_{X^\tau}$  given by the composition of the respective trivializations. If  $\kappa(\mathcal{E}_1, \Theta_1) = \kappa(\mathcal{E}_2, \Theta_2)$  Lemma 4.6 assures that there is an  $\mathfrak{R}$ -isomorphism  $\psi : \det(\mathcal{E}_1) \rightarrow \det(\mathcal{E}_2)$  such that  $\det(\Psi) = \psi|_{X^\tau}$ . This allows to construct the fiber bundle  $\mathcal{S}\mathcal{U}(\mathcal{E}_1, \mathcal{E}_2, \psi)$  and  $\Psi$  provides an equivariant section of the restriction  $\mathcal{S}\mathcal{U}(\mathcal{E}_1, \mathcal{E}_2, \psi)|_{X^\tau}$ . Now, the free  $\mathbb{Z}_2$ -celles of  $X$  have dimensions less than 4 and  $\pi_j(\mathcal{S}\mathcal{U}(2)) = 0$  for  $j = 0, 1, 2$  (equivalently  $\mathcal{S}\mathcal{U}(2)$  is 2-connected). These facts are sufficient to build a global section  $\Upsilon : X \rightarrow \mathcal{S}\mathcal{U}(\mathcal{E}_1, \mathcal{E}_2, \psi)$  which extends  $\Psi$  in the sense that  $\Upsilon|_{X^\tau} = \Psi$ . The explicit construction is realized following the same arguments in the proof of [DG1, Proposition 4.23]. The existence of  $\Upsilon$  implies the  $\mathfrak{Q}$ -isomorphism  $(\mathcal{E}_1, \Theta_1) \simeq (\mathcal{E}_2, \Theta_2)$  and this proves that  $\kappa$  is injective. The group structure on  $\text{Vec}_{\mathfrak{Q}}^{2m}(X, \tau)$ , which makes  $\kappa$  a group homomorphism, has been described in full detail in the proof of [DG2, Theorem 1.1] and is based on the splitting (4.4) and on the additivity of the FKMM-invariant with respect to the Whitney sum (cf. property (4) in Section 2.5).

The proof in the case (2) is quite similar. Let  $(\mathcal{E}_1, \Theta_1)$  and  $(\mathcal{E}_2, \Theta_2)$  be two “Quaternionic” vector bundles of rank 2 over  $(X, \tau)$  and assume that  $\kappa(\mathcal{E}_1, \Theta_1) = \kappa(\mathcal{E}_2, \Theta_2)$  in  $H_{\mathbb{Z}_2}^2(X, \mathbb{Z}(1))$ . This implies that there is an isomorphism of “Real” line bundles  $\psi : \det(\mathcal{E}_1) \rightarrow \det(\mathcal{E}_2)$  which allows to build the fiber bundle  $\mathcal{S}\mathcal{U}(\mathcal{E}_1, \mathcal{E}_2, \psi)$ . Since  $X$  has free  $\mathbb{Z}_2$ -celles of dimension less than 4 and the typical fiber  $\mathcal{S}\mathcal{U}(2)$  is 2-connected it follows that  $\mathcal{S}\mathcal{U}(\mathcal{E}_1, \mathcal{E}_2, \psi)$  admits a global section which provides a  $\mathfrak{Q}$ -isomorphism between  $(\mathcal{E}_1, \Theta_1)$  and  $(\mathcal{E}_2, \Theta_2)$ . The group structure can be introduced as in the case (i). The last claim is a consequence of Lemma 4.4.  $\blacksquare$

The odd rank case requires a base space with a free involution  $X^\tau = \emptyset$  and it is completely determined by the nature of  $\text{Pic}_{\mathfrak{Q}}(X, \tau)$ . More precisely, when  $\text{Pic}_{\mathfrak{Q}}(X, \tau) = \emptyset$  is not possible to build “Quaternionic” vector bundles (cf. Lemma 4.4) and the  $\kappa$ -invariant simply cannot be defined. The case  $\text{Pic}_{\mathfrak{Q}}(X, \tau) \neq \emptyset$  is described in the next result.

**Theorem 4.8** (Injectivity: odd rank case). *Let  $(X, \tau)$  be a low dimensional involutive space in the sense of Definition 4.1. Assume  $X^\tau = \emptyset$  and  $\text{Pic}_{\mathfrak{Q}}(X, \tau) \neq \emptyset$ . Then, there is a bijection*

$$\kappa : \text{Vec}_{\mathfrak{Q}}^{2m-1}(X, \tau) \xrightarrow{\simeq} H_{\mathbb{Z}_2}^2(X, \mathbb{Z}(1)), \quad m \in \mathbb{N}$$

*and the form of  $\kappa$  depends on a normalization given by the choice a reference element  $[\mathcal{L}_{\text{ref}}] \in \text{Pic}_{\mathfrak{Q}}(X, \tau)$ .*

*Proof.* By combining equation (4.9) with Corollary 3.2 one obtains the bijections

$$\mathrm{Vec}_{\mathbb{Q}}^{2m-1}(X, \tau) \simeq \mathrm{Pic}_{\mathbb{Q}}(X, \tau) \simeq H_{\mathbb{Z}_2}^2(X, \mathbb{Z}(1)), \quad m \in \mathbb{N}$$

both depending on the election of a reference element  $[\mathcal{L}_{\mathrm{ref}}] \in \mathrm{Pic}_{\mathbb{Q}}(X, \tau)$ . More precisely, one has the isomorphisms

$$[\mathcal{E}_{\mathbb{Q}}] \simeq \begin{cases} [\mathcal{L}_{\mathrm{ref}}] \otimes [X \times \mathbb{C}]^{\oplus(2m-1)} \simeq [\mathcal{L}_{\mathrm{ref}}]^{\oplus(2m-2)} \oplus [\mathcal{L}_{\mathrm{ref}}] & d = 0, 1 \\ [\mathcal{L}_{\mathrm{ref}}] \otimes ([X \times \mathbb{C}]^{\oplus(2m-2)} \oplus \mathcal{L}_{\mathbb{R}}) \simeq [\mathcal{L}_{\mathrm{ref}}]^{\oplus(2m-2)} \oplus [\mathcal{L}_{\mathrm{ref}} \otimes \mathcal{L}_{\mathbb{R}}] & d = 2, 3 \end{cases}$$

which induce the one-to-one identifications

$$[\mathcal{E}_{\mathbb{Q}}] \mapsto [\mathcal{L}_{\mathrm{ref}} \otimes \mathcal{L}_{\mathbb{R}}] \mapsto [\mathcal{L}_{\mathbb{R}}] \mapsto c_1^{\mathbb{R}}(\mathcal{L}_{\mathbb{R}})$$

with the convention that  $\mathcal{L}_{\mathbb{R}}$  is automatically trivial when  $d = 0, 1$ . The value of the FKMM-invariant can be fixed in accordance to Definition 3.3 by the prescription

$$\kappa(\mathcal{E}_{\mathbb{Q}}) = c_1^{\mathbb{R}}(\mathcal{L}_{\mathbb{R}})$$

which implies the normalization  $\kappa(\mathcal{L}_{\mathrm{ref}}^{\oplus(2m-1)}) = 0$ . ■

**4.3. The question of the surjectivity of the FKMM-invariant.** Theorem 4.7 and Theorem 4.8 assure the injectivity of the FKMM-invariant  $\kappa$  in the case of low dimensional base spaces ( $d \leq 3$ ). On the one hand, this is enough to distinguish between different topological phases. On the other hand it could be valuable to know whether the cohomology group where  $\kappa$  maps is a complete set of invariants or if it is redundant. It turns out that, for a big class low dimensional spaces the FKMM-invariant provides bijections. For instance, this is the case for the involutive spheres  $\mathbb{S}^{p,q}$  and the involutive tori  $\mathbb{T}^{a,b,c}$ , discussed in Section 4.4 and Section 4.5 respectively, or for oriented surfaces with finitely many isolated fixed points [DG2, Corollary 4.2]. Therefore, one is legitimate to ask whether the surjectivity in low dimension is a general property of the FKMM-invariant or not!

Let us start with two observation: ( $\alpha$ ) The FKMM-invariant fails to be surjective in dimension bigger than three where other invariants like the second Chern class start to play a role in the classification (cf. [DG2, Theorem 1.4]); ( $\beta$ ) The odd rank case (which requires  $X^\tau = \emptyset$ ) is completely specified by the nature of  $\mathrm{Pic}_{\mathbb{Q}}(X, \tau)$  as proved in Theorem 4.8. Therefore, the question of the surjectivity is relevant only for even rank “Quaternionic” vector bundles over low dimensional involutive spaces. We point out that the study of this question deserves a separated analysis for the cases  $X^\tau = \emptyset$  and  $X^\tau \neq \emptyset$ .

The study of the surjectivity of  $\kappa$  goes beyond the scope of this work. However, we can anticipate some results which will appear on a separate work devoted to the analysis of this problem [DG5]. Let  $(X, \tau)$  be an involutive space of dimension  $d$  which fulfills Assumption 2.1. Then:

- (1) When  $d = 0, 1$  the  $\kappa$ -invariant provides a “trivial” isomorphism.
- (2) When  $d = 2$  the  $\kappa$ -invariant provides isomorphisms

$$\mathrm{Vec}_{\mathbb{Q}}^{2m}(X, \tau) \stackrel{\kappa}{\simeq} \begin{cases} H_{\mathbb{Z}_2}^2(X, \mathbb{Z}(1)) & \text{if } X^\tau = \emptyset \\ H_{\mathbb{Z}_2}^2(X|X^\tau, \mathbb{Z}(1)) & \text{if } X^\tau \neq \emptyset. \end{cases}$$

- (3) When  $d = 3$  the  $\kappa$ -invariant *fails* to be surjective.

The proof of (1) amounts to show that the condition  $d = 0, 1$  forces  $H_{\mathbb{Z}_2}^2(X, \mathbb{Z}(1)) = 0$  for the free involution case  $X^\tau = \emptyset$  and  $H_{\mathbb{Z}_2}^2(X|X^\tau, \mathbb{Z}(1)) = 0$  when  $X^\tau \neq \emptyset$ . The proof of (2) is quite technical. The payoff of (1) and (2) is that one can see the  $\kappa$ -invariant as the generalization of the isomorphisms (1.4) and (1.6) for the “Quaternionic” category up to dimension two. The claim (3) marks the difference between the FKMM-invariant and the first (“Real”) Chern class.

The proof of (3) is based on the construction of an explicit counterexample Let  $\mathbb{S}^3 := \{(z_0, z_1) \in \mathbb{C}^2 \mid |z_0|^2 + |z_1|^2 = 1\} \subset \mathbb{C}^2$  be the three dimensional sphere viewed as the unit sphere in  $\mathbb{C}^2$ . Consider the action of  $\mathbb{Z}_4 \simeq \{\pm 1, \pm i\}$  on  $\mathbb{S}^3$  given by  $(z_0, z_1) \mapsto (\rho z_0, \rho z_1)$  for all  $\rho \in \mathbb{Z}_4$ . The quotient space

$$L(1; 4) := \mathbb{S}^3 / \mathbb{Z}_4$$

is called (3-dimensional) *lens space* (cf. [BT, Example 18.5] or [Hat, Example 2.43] for more details). Since  $\mathbb{Z}_4$  is preserved by the complex conjugation (with our convention) the involution on  $\mathbb{S}^3 \subset \mathbb{C}^2$  induced by  $(z_0, z_1) \mapsto (\overline{z_0}, \overline{z_1})$  descends to an involution  $\tau$  on  $L(1; 4)$ . The involutive space  $(L(1; 4), \tau)$  is a smooth (3-dimensional) manifold with a smooth involution, hence it admits a  $\mathbb{Z}_2$ -CW-complex structure [May, Theorem 3.6]. Moreover this space has a fixed point set of the type  $L(1; 4)^\tau \simeq \mathbb{S}^1 \sqcup \mathbb{S}^1$ . One can construct the “Quaternionic” vector bundles over  $(L(1; 4), \tau)$  by the *clutching construction* providing

$$\text{Vec}_{\mathbb{Q}}^{2m}(L(1; 4), \tau) = \mathbb{Z}_4. \quad (4.11)$$

On the other hand the computation of the equivariant cohomology provides

$$H_{\mathbb{Z}_2}^2(L(1; 4)|L(1; 4)^\tau, \mathbb{Z}(1)) = \mathbb{Z}_8. \quad (4.12)$$

The proofs of (4.11) and (4.12) are quite elaborate and technical and will be presented in [DG5]. In any case, the important message conveyed by (4.11) and (4.12) is that the map

$$\kappa : \text{Vec}_{\mathbb{Q}}^{2m}(L(1; 4), \tau) \longrightarrow H_{\mathbb{Z}_2}^2(L(1; 4)|L(1; 4)^\tau, \mathbb{Z}(1))$$

cannot be surjective, confirming the claim (3).

**4.4. Classification over low dimensional involutive spheres.** In this section we apply the results of Section 4.1 and Section 4.2 to provide the complete classification of “Quaternionic” vector bundles over the involutive spheres of type  $\mathbb{S}^{p,q}$  in low dimension  $d = p + q - 1 \leq 3$ . These results have been summarized in Table 1.1. Since  $\mathbb{S}^{0,0} = \emptyset$  we conventionally write

$$\text{Vec}_{\mathbb{Q}}^k(\mathbb{S}^{0,0}) = \emptyset.$$

• **Spheres of type  $\mathbb{S}^{p,0}$ .** These are the spheres with the trivial involution which fixes all points. The isomorphism [DG2, Proposition 2.2]

$$\text{Vec}_{\mathbb{Q}}^{2m}(\mathbb{S}^{p,0}) \simeq \text{Vec}_{\mathbb{H}}^m(\mathbb{S}^{p-1})$$

and the stable rank condition for  $\mathbb{H}$ -vector bundles [Hus, Chapter 9, Theorem 1.2] provide

$$\text{Vec}_{\mathbb{Q}}^{2m}(\mathbb{S}^{p,0}) = 0, \quad 1 \leq p \leq 4.$$

• **Spheres of type  $\mathbb{S}^{0,q}$ .** These are the spheres with free antipodal involution and are the only spheres which admit “Quaternionic” vector bundles of odd rank. The line bundle case has been already discussed in Section 3.2.

[q=1,2] - When  $q = 1$  the sphere  $\mathbb{S}^{0,1}$  is the two points set  $\{-1, +1\}$  endowed with the flip involution  $\pm 1 \mapsto \mp 1$  and one has that

$$\text{Vec}_{\mathbb{Q}}^k(\mathbb{S}^{0,1}) = 0, \quad \forall k \in \mathbb{N}.$$

For  $k = 2m$  the unique element is given by the (trivial) “Quaternionic” product bundle over  $\{-1, +1\}$  while for  $k = 2m + 1$  the unique element is given by the direct sum of a rank  $2m$  “Quaternionic” trivial bundle with the Dupont “Quaternionic” line bundle described in Example 3.5. Also for  $q = 2$  one has

$$\text{Vec}_{\mathbb{Q}}^k(\mathbb{S}^{0,2}) = 0, \quad \forall k \in \mathbb{N}.$$

When  $k = 2m$  the result is justified by equation (4.3) and the unique element represented by the trivial “Quaternionic” product bundle. When  $k = 2m + 1$  the result is justified by (4.9) and Example 3.7. More precisely, in the odd rank case the unique element is represented by the direct sum of a trivial “Quaternionic” vector bundle of rank  $2m$  with the “Quaternionic” line bundle  $\mathcal{L}_0$  described in Example 3.7.

[q=3] - In the odd rank case equation (4.9) and the Example 3.8 provide

$$\text{Vec}_{\mathbb{Q}}^{2m-1}(\mathbb{S}^{0,3}) \simeq \text{Pic}_{\mathbb{Q}}(\mathbb{S}^{0,3}) \simeq 2\mathbb{Z} + 1, \quad \forall m \in \mathbb{N}$$

where the (second) identification is given by the first Chern class  $c_1$ . This identification requires a reference element in  $\text{Pic}_{\mathbb{Q}}(\mathbb{S}^{0,3})$  which can be chosen to be the line bundle  $\mathcal{L}_1$  constructed in Example 3.8. The proof of Theorem 4.8 shows that each  $[\mathcal{E}_{\mathbb{Q}}] \in \text{Vec}_{\mathbb{Q}}^{2m-1}(\mathbb{S}^{0,3})$  can be uniquely represented

by an element  $[\mathcal{L}_{\mathfrak{R}}] \in \text{Pic}_{\mathfrak{R}}(\mathbb{S}^{0,3})$  according to the factorization  $[\mathcal{E}_{\mathfrak{Q}}] \simeq [\mathcal{L}_1]^{\oplus(2m-2)} \oplus [\mathcal{L}_1 \otimes \mathcal{L}_{\mathfrak{R}}]$ . Accordingly, one can fix the FKMM-invariant by the prescription

$$\kappa(\mathcal{E}_{\mathfrak{Q}}) := c_1^{\mathfrak{R}}(\mathcal{L}_{\mathfrak{R}}).$$

The identification  $2c_1^{\mathfrak{R}}(\mathcal{L}_{\mathfrak{R}}) \simeq c_1(\mathcal{L}_{\mathfrak{R}})$  between the “Real” Chern class and the standard Chern class of  $\mathcal{L}_{\mathfrak{R}}$  discussed in Example 3.8 provides

$$c_1(\mathcal{E}_{\mathfrak{Q}}) \simeq (2m-2) c_1(\mathcal{L}_1) + c_1(\mathcal{L}_1) + c_1(\mathcal{L}_{\mathfrak{R}}) \simeq (2m-1) c_1(\mathcal{L}_1) + 2 c_1^{\mathfrak{R}}(\mathcal{L}_{\mathfrak{R}})$$

Since  $c_1(\mathcal{L}_1) = 1$  and  $c_1^{\mathfrak{R}}(\mathcal{L}_{\mathfrak{R}}) \in \mathbb{Z}$  one obtains that the Chern class of any representative in  $\text{Vec}_{\mathfrak{Q}}^{2m-1}(\mathbb{S}^{0,3})$  is necessarily odd and it is sufficient for a complete characterization of the isomorphism class  $[\mathcal{E}_{\mathfrak{Q}}]$ . Moreover, the FKMM-invariant takes the form

$$\kappa(\mathcal{E}_{\mathfrak{Q}}) = \frac{c_1(\mathcal{E}_{\mathfrak{Q}}) - (2m-1)}{2}, \quad [\mathcal{E}_{\mathfrak{Q}}] \in \text{Vec}_{\mathfrak{Q}}^{2m-1}(\mathbb{S}^{0,3}).$$

In the even rank case one has

$$\text{Vec}_{\mathfrak{Q}}^{2m}(\mathbb{S}^{0,3}) \simeq \text{Vec}_{\mathfrak{Q}}^2(\mathbb{S}^{0,3}) \xrightarrow{\kappa} H_{\mathbb{Z}_2}^2(\mathbb{S}^{0,3}, \mathbb{Z}(1)) \simeq \mathbb{Z}$$

where the first isomorphism is justified by (4.4) and the FKMM-invariant provides an injective map according to Theorem 4.7 (2). Moreover, the existence of  $\mathcal{L}_1$  assures that  $\kappa$  is indeed an isomorphism. More precisely, for each  $[\mathcal{E}_{\mathfrak{Q}}] \in \text{Vec}_{\mathfrak{Q}}^{2m}(\mathbb{S}^{0,3})$  there exists a unique  $[\mathcal{E}_{\mathfrak{R}}] \in \text{Vec}_{\mathfrak{R}}^{2m}(\mathbb{S}^{0,3})$  such that  $[\mathcal{E}_{\mathfrak{Q}}] \simeq [\mathcal{L}_1 \otimes \mathcal{E}_{\mathfrak{R}}]$ . The determinant construction provides  $\det(\mathcal{E}_{\mathfrak{Q}}) = \mathcal{L}_1^{\otimes 2m} \otimes \det(\mathcal{E}_{\mathfrak{R}})$  and a straightforward computation shows that

$$c_1(\mathcal{E}_{\mathfrak{Q}}) \simeq c_1(\det(\mathcal{E}_{\mathfrak{Q}})) \simeq 2m c_1(\mathcal{L}_1) + c_1(\det(\mathcal{E}_{\mathfrak{R}})) \simeq 2m c_1(\mathcal{L}_1) + 2 c_1^{\mathfrak{R}}(\det(\mathcal{E}_{\mathfrak{R}})).$$

Since  $c_1(\mathcal{L}_1) = 1$  and  $c_1^{\mathfrak{R}}(\mathcal{L}_{\mathfrak{R}}) \in \mathbb{Z}$  one concludes that the Chern class of any representative in  $\text{Vec}_{\mathfrak{Q}}^{2m}(\mathbb{S}^{0,3})$  must be even. Moreover, by definition one has

$$\kappa(\mathcal{E}_{\mathfrak{Q}}) := c_1^{\mathfrak{R}}(\det(\mathcal{E}_{\mathfrak{Q}})) = \frac{c_1(\mathcal{E}_{\mathfrak{Q}})}{2}, \quad [\mathcal{E}_{\mathfrak{Q}}] \in \text{Vec}_{\mathfrak{Q}}^{2m}(\mathbb{S}^{0,3}).$$

Summarizing, we proved that

$$\begin{cases} \text{Vec}_{\mathfrak{Q}}^{2m}(\mathbb{S}^{0,3}) \xrightarrow{c_1} 2\mathbb{Z} \\ \text{Vec}_{\mathfrak{Q}}^{2m-1}(\mathbb{S}^{0,3}) \xrightarrow{c_1} 2\mathbb{Z} + 1 \end{cases}, \quad \forall m \in \mathbb{N}. \quad (4.13)$$

We point out that the first Chern class  $c_1$  turns out to be a complete invariant for the classification of  $\mathfrak{Q}$ -bundles of any rank over  $\mathbb{S}^{0,3}$ . The (4.13) recovers the results recently presented in [GR].

[q=4] - By combining Lemma 4.4 and Example 3.6 one concludes

$$\text{Vec}_{\mathfrak{Q}}^{2m-1}(\mathbb{S}^{0,4}) = \emptyset, \quad \forall m \in \mathbb{N}$$

for the odd rank case. For the even rank case one has

$$\text{Vec}_{\mathfrak{Q}}^{2m}(\mathbb{S}^{0,4}) \simeq \text{Vec}_{\mathfrak{Q}}^2(\mathbb{S}^{0,4}) \xrightarrow{\kappa} H_{\mathbb{Z}_2}^2(\mathbb{S}^{0,4}, \mathbb{Z}(1)) \simeq 0$$

where the first isomorphism is given by (4.4). The injectivity of the  $\kappa$ -invariant proved in Theorem 4.7 (2) implies

$$\text{Vec}_{\mathfrak{Q}}^{2m}(\mathbb{S}^{0,4}) = 0, \quad \forall m \in \mathbb{N}$$

and the unique element is represented by the rank  $2m$  trivial  $\mathfrak{Q}$ -bundle.

• **Spheres of type  $\mathbb{S}^{1,q}$ .** These are the spheres with TR-involution (two distinguished fixed points) which have been studied in [DG2]. We recall that:

$$\text{Vec}_{\mathfrak{Q}}^{2m}(\mathbb{S}^{1,0}) \simeq \text{Vec}_{\mathfrak{Q}}^{2m}(\mathbb{S}^{1,1}) = 0, \quad \text{Vec}_{\mathfrak{Q}}^{2m}(\mathbb{S}^{1,2}) \simeq \text{Vec}_{\mathfrak{Q}}^{2m}(\mathbb{S}^{1,3}) \simeq \mathbb{Z}_2.$$

In the non-trivial cases  $\mathbb{S}^{1,2}$  and  $\mathbb{S}^{1,3}$  the  $\mathbb{Z}_2$ -phase is discriminated by the FKMM-invariant.

• **Spheres of type  $\mathbb{S}^{2,q}$ .** In this case the fixed point set  $(\mathbb{S}^{2,q})^\theta \simeq \mathbb{S}^1$  is a circle and only even rank “Quaternionic” vector bundles are possible. Equation (4.4) and Theorem 4.7 (1) provide

$$\mathrm{Vec}_{\mathbb{Q}}^{2m}(\mathbb{S}^{2,q}) \simeq \mathrm{Vec}_{\mathbb{Q}}^2(\mathbb{S}^{2,q}) \xrightarrow{\kappa} H_{\mathbb{Z}_2}^2(\mathbb{S}^{2,q}|\mathbb{S}^1, \mathbb{Z}(1)) \quad q = 1, 2 \quad \forall m \in \mathbb{N}$$

with  $\kappa$  being an injection.

[q=1] - Proposition B.3 provides

$$\mathrm{Vec}_{\mathbb{Q}}^2(\mathbb{S}^{2,1}) \xrightarrow{\kappa} H_{\mathbb{Z}_2}^2(\mathbb{S}^{2,1}|\mathbb{S}^1, \mathbb{Z}(1)) \simeq 2\mathbb{Z}$$

Moreover, one has that the isomorphism  $H_{\mathbb{Z}_2}^2(\mathbb{S}^{2,1}|\mathbb{S}^1, \mathbb{Z}(1)) \simeq 2\mathbb{Z}$  is induced by the inclusion

$$H_{\mathbb{Z}_2}^2(\mathbb{S}^{2,1}|\mathbb{S}^1, \mathbb{Z}(1)) \xrightarrow{\delta_2} H_{\mathbb{Z}_2}^2(\mathbb{S}^{2,1}, \mathbb{Z}(1)) \xrightarrow{f} H^2(\mathbb{S}^2, \mathbb{Z}) \xrightarrow{c_1} \mathbb{Z}$$

by means of the first Chern class of the underlying complex vector bundle. The surjectivity of  $\kappa$  can be proved as follow: Let  $\mathcal{E}_0$  be the trivial element in  $\mathrm{Vec}_{\mathbb{Q}}^2(\mathbb{S}^{2,1})$  and  $\mathcal{L}_1 \in \mathrm{Pic}_{\mathbb{R}}(\mathbb{S}^{2,1})$  such that  $c_1(\mathcal{L}_1)=1$ . An  $\mathbb{R}$ -line bundle with this property can be explicitly constructed as showed in the proof of Proposition B.3. Since  $c_1(\mathcal{E}_0) = 0$  the “Quaternionic” vector bundle  $\mathcal{E}_{2n} := \mathcal{E}_0 \otimes (\mathcal{L}_1^{\otimes n})$  has Chern class

$$c_1(\mathcal{E}_{2n}) = \mathrm{rk}(\mathcal{E}_0) c_1(\mathcal{L}_1^{\otimes n}) = 2n c_1(\mathcal{L}_1) = 2n .$$

This fact shows that  $\kappa$  is also surjective. In particular, we proved the isomorphism

$$\mathrm{Vec}_{\mathbb{Q}}^{2m}(\mathbb{S}^{2,1}) \xrightarrow{c_1} 2\mathbb{Z}$$

which says that a vector bundle over  $\mathbb{S}^{2,1}$  can be endowed with a unique  $\mathbb{Q}$ -structure if and only if its first Chern class is even.

[q=2] - Equation (B.5) shows that  $H_{\mathbb{Z}_2}^2(\mathbb{S}^{2,2}|\mathbb{S}^1, \mathbb{Z}(1)) = 0$ , hence the injectivity of  $\kappa$  immediately implies that  $\mathrm{Vec}_{\mathbb{Q}}^{2m}(\mathbb{S}^{2,2}) = 0$  for all  $m \in \mathbb{N}$ .

• **Spheres of type  $\mathbb{S}^{3,q}$ .** The only case of interest in low dimension is  $\mathbb{S}^{3,1}$ . This space contains a two-dimensional sphere as fixed point set  $(\mathbb{S}^{3,1})^\theta \simeq \mathbb{S}^2$ . Therefore, only the even rank case is relevant. Equation (4.4) and Theorem 4.7 (1) provide

$$\mathrm{Vec}_{\mathbb{Q}}^{2m}(\mathbb{S}^{3,1}) \simeq \mathrm{Vec}_{\mathbb{Q}}^2(\mathbb{S}^{3,1}) \xrightarrow{\kappa} H_{\mathbb{Z}_2}^2(\mathbb{S}^{3,1}|\mathbb{S}^2, \mathbb{Z}(1))$$

with  $\kappa$  being an injection. Since equation (B.6) shows that  $H_{\mathbb{Z}_2}^2(\mathbb{S}^{3,1}|\mathbb{S}^2, \mathbb{Z}(1)) = 0$  one concludes that  $\mathrm{Vec}_{\mathbb{Q}}^{2m}(\mathbb{S}^{3,1}) = 0$  for all  $m \in \mathbb{N}$ .

**4.5. Classification over low dimensional involutive tori.** This section is devoted to the complete classification of “Quaternionic” vector bundles over involutive tori of the type  $\mathbb{T}^{a,b,c}$  in low dimension  $a + b + c \leq 3$ . These results has been summarized in Table 1.2 and Table 1.3. Since  $\mathbb{T}^{0,0,0} = \emptyset$  we conventionally write

$$\mathrm{Vec}_{\mathbb{Q}}^k(\mathbb{T}^{0,0,0}) = \emptyset .$$

#### - Cases with fixed points -

Involutive tori of type  $\mathbb{T}^{a,b,0}$  have non-empty fixed point sets. There are three different two-dimensional tori and four different three-dimensional tori of this type. For all these cases only even rank  $\mathbb{Q}$ -bundles are possible due to the presence of fixed points.

• **The torus  $\mathbb{T}^{a,0,0}$ .** Spaces of type  $\mathbb{T}^{a,0,0}$  are  $a$ -dimensional tori with the trivial involution which fixes all points. According to [DG2, Proposition 2.2] one has the isomorphism

$$\mathrm{Vec}_{\mathbb{Q}}^{2m}(\mathbb{T}^{a,0,0}) \simeq \mathrm{Vec}_{\mathbb{H}}^m(\mathbb{T}^a) .$$

The stable rank condition for  $\mathbb{H}$ -vector bundles [Hus, Chapter 9, Theorem 1.2] provides

$$\mathrm{Vec}_{\mathbb{Q}}^{2m}(\mathbb{T}^{a,0,0}) = 0 , \quad 1 \leq a \leq 3 \quad \forall m \in \mathbb{N} .$$



- **The torus  $\mathbb{T}^{0,b,0}$ .** The spaces  $\mathbb{T}^{0,b,0}$  are  $b$ -dimensional tori with the TR-involution ( $2^b$  distinguished fixed points) which have been studied in [DG2]. We recall that:

$$\mathrm{Vec}_{\mathfrak{Q}}^{2m}(\mathbb{T}^{0,2,0}) \simeq \mathbb{Z}_2, \quad \mathrm{Vec}_{\mathfrak{Q}}^{2m}(\mathbb{T}^{0,3,0}) \simeq \mathbb{Z}_2^4 \quad \mathrm{Vec}_{\mathfrak{Q}}^{2m}(\mathbb{T}^{0,4,0}) \simeq \mathbb{Z}_2^{10} \oplus \mathbb{Z}.$$

for all  $m \in \mathbb{N}$ . The  $\mathbb{Z}_2$ -phases are discriminated by the FKMM-invariant.

- **The torus  $\mathbb{T}^{1,1,0}$ .** In this case the fixed point set  $(\mathbb{T}^{1,1,0})^\tau \simeq \mathbb{S}^1 \sqcup \mathbb{S}^1$  is given by a disjoint union of two circles. Equation (4.4) and Theorem 4.7 (1) provide

$$\mathrm{Vec}_{\mathfrak{Q}}^{2m}(\mathbb{T}^{1,1,0}) \simeq \mathrm{Vec}_{\mathfrak{Q}}^2(\mathbb{T}^{1,1,0}) \xrightarrow{\kappa} H_{\mathbb{Z}_2}^2(\mathbb{T}^{1,1,0}|(\mathbb{T}^{1,1,0})^\tau, \mathbb{Z}(1)) \simeq 2\mathbb{Z}.$$

with  $\kappa$  being an injection. The last isomorphism is proved in Proposition C.2 and is induced by the composition

$$H_{\mathbb{Z}_2}^2(\mathbb{T}^{1,1,0}|(\mathbb{T}^{1,1,0})^\tau, \mathbb{Z}(1)) \xrightarrow{\delta_2} H_{\mathbb{Z}_2}^2(\mathbb{T}^{1,1,0}, \mathbb{Z}(1)) \xrightarrow{f} H^2(\mathbb{T}^2, \mathbb{Z}) \xrightarrow{c_1} \mathbb{Z}$$

where the forgetting map  $f$  turns out to be surjective. The surjectivity of  $\kappa$  can be verified as follow: Let  $\mathcal{E}_0$  be the trivial element in  $\mathrm{Vec}_{\mathfrak{Q}}^2(\mathbb{T}^{1,1,0})$  and  $\mathcal{L}_{1,1} \in \mathrm{Pic}_{\mathfrak{R}}(\mathbb{T}^{1,1,0})$  the non trivial  $\mathfrak{R}$ -line bundle with  $c_1(\mathcal{L}_{1,1}) = 1$  constructed in the proof of Proposition C.2. Since  $c_1(\mathcal{E}_0) = 0$  the  $\mathfrak{Q}$ -bundle  $\mathcal{E}_{2n} := \mathcal{E}_0 \otimes (\mathcal{L}_{1,1}^{\otimes n})$  has Chern class

$$c_1(\mathcal{E}_{2n}) = \mathrm{rk}(\mathcal{E}_0) c_1(\mathcal{L}_{1,1}^{\otimes n}) = 2n c_1(\mathcal{L}_{1,1}) = 2n.$$

This fact proves that  $\kappa$  is also surjective and in turn one has the isomorphism

$$\mathrm{Vec}_{\mathfrak{Q}}^{2m}(\mathbb{T}^{1,1,0}) \xrightarrow{c_1} 2\mathbb{Z}, \quad \forall m \in \mathbb{N}.$$

In particular a vector bundle over  $\mathbb{T}^{1,1,0}$  can be endowed with a unique  $\mathfrak{Q}$ -structure if and only if it has an even Chern class. A similar result has been recently obtained in [GR].

- **The torus  $\mathbb{T}^{2,1,0}$ .** The fixed point set  $(\mathbb{T}^{2,1,0})^\tau \simeq \mathbb{T}^2 \sqcup \mathbb{T}^2$  consists of two disjoint copies of a two-dimensional torus. One has

$$\mathrm{Vec}_{\mathfrak{Q}}^{2m}(\mathbb{T}^{2,1,0}) \simeq \mathrm{Vec}_{\mathfrak{Q}}^2(\mathbb{T}^{2,1,0}) \xrightarrow{\kappa} H_{\mathbb{Z}_2}^2(\mathbb{T}^{2,1,0}|(\mathbb{T}^{2,1,0})^\tau, \mathbb{Z}(1)) \simeq (2\mathbb{Z})^2$$

where the first isomorphism is justified in (4.4) and Theorem 4.7 (1) assures that  $\kappa$  is an injection. The last isomorphism is proved in Proposition C.3 and is induced by the composition

$$H_{\mathbb{Z}_2}^2(\mathbb{T}^{2,1,0}|(\mathbb{T}^{2,1,0})^\tau, \mathbb{Z}(1)) \xrightarrow{\delta_2} H_{\mathbb{Z}_2}^2(\mathbb{T}^{2,1,0}, \mathbb{Z}(1)) \xrightarrow{f} H^2(\mathbb{T}^3, \mathbb{Z}) \xrightarrow{c_1} \mathbb{Z}^3$$

where the forgetting map  $f$  turns out to be bijective between the free part of  $H_{\mathbb{Z}_2}^2(\mathbb{T}^{2,1,0}, \mathbb{Z}(1))$  and a  $\mathbb{Z}^2$ -summand in  $H^2(\mathbb{T}^3, \mathbb{Z})$ . Let  $\mathcal{E}_2 \rightarrow \mathbb{T}^{1,1,0}$  be the element of  $\mathrm{Vec}_{\mathfrak{Q}}^{2m}(\mathbb{T}^{1,1,0})$  identified by the Chern class  $c_1(\mathcal{E}_2) = 2$  and consider the two pullbacks of  $\mathcal{E}_2$  in  $\mathrm{Vec}_{\mathfrak{Q}}^{2m}(\mathbb{T}^{2,1,0})$  under the two distinct projections  $\mathbb{T}^{2,1,0} \rightarrow \mathbb{T}^{1,1,0}$ . These two vector bundles provide a basis for the bijection

$$\mathrm{Vec}_{\mathfrak{Q}}^{2m}(\mathbb{T}^{2,1,0}) \xrightarrow{c_1} (\mathbb{Z}^2)^2, \quad \forall m \in \mathbb{N}$$

showing in turn that  $\kappa$  is bijective. In particular one has that a vector bundle over  $\mathbb{T}^{2,1,0}$  can be endowed with a unique  $\mathfrak{Q}$ -structure if and only if it has a Chern class of type  $(2n, 2n', 0) \in \mathbb{Z}^3$ .

- **The torus  $\mathbb{T}^{1,2,0}$ .** In this case the fixed point set  $(\mathbb{T}^{1,2,0})^\tau \simeq \mathbb{S}^1 \sqcup \mathbb{S}^1 \sqcup \mathbb{S}^1 \sqcup \mathbb{S}^1$  is the disjoint union of four circles. From (4.4) and Theorem 4.7 (1) one obtains

$$\mathrm{Vec}_{\mathfrak{Q}}^{2m}(\mathbb{T}^{1,2,0}) \simeq \mathrm{Vec}_{\mathfrak{Q}}^2(\mathbb{T}^{1,2,0}) \xrightarrow{\kappa} H_{\mathbb{Z}_2}^2(\mathbb{T}^{1,2,0}|(\mathbb{T}^{1,2,0})^\tau, \mathbb{Z}(1)) \simeq \mathbb{Z}_2 \oplus (2\mathbb{Z})^2$$

with  $\kappa$  being an injection. The last isomorphism, proved in Proposition C.4, enters in the composition of maps

$$H_{\mathbb{Z}_2}^2(\mathbb{T}^{1,2,0}|(\mathbb{T}^{1,2,0})^\tau, \mathbb{Z}(1)) \xrightarrow{\delta_2} H_{\mathbb{Z}_2}^2(\mathbb{T}^{1,2,0}, \mathbb{Z}(1)) \xrightarrow{f} H^2(\mathbb{T}^3, \mathbb{Z}) \xrightarrow{c_1} \mathbb{Z}^3$$

where  $\delta_2$  acts injectively on the free part of  $H_{\mathbb{Z}_2}^2(\mathbb{T}^{1,2,0}|(\mathbb{T}^{1,2,0})^\tau, \mathbb{Z}(1))$  and the forgetting map  $f$  is a bijection between the free part of  $H_{\mathbb{Z}_2}^2(\mathbb{T}^{1,2,0}, \mathbb{Z}(1))$  and a  $\mathbb{Z}^2$ -summand in  $H^2(\mathbb{T}^3, \mathbb{Z})$ . It turn out that

the map  $\kappa$  is also surjectivity. In fact, the  $\mathbb{Z}_2$ -summand in  $\text{Vec}_{\mathbb{Q}}^{2m}(\mathbb{T}^{1,2,0})$  is generated by the pullback of  $\text{Vec}_{\mathbb{Q}}^{2m}(\mathbb{T}^{0,2,0}) \simeq \mathbb{Z}_2$  under the projection  $\mathbb{T}^{1,2,0} \rightarrow \mathbb{T}^{0,2,0}$ . The two  $2\mathbb{Z}$ -summands, instead, are generated by the pullback of  $\text{Vec}_{\mathbb{Q}}^{2m}(\mathbb{T}^{1,1,0}) \simeq 2\mathbb{Z}$  under the two distinct projections  $\mathbb{T}^{1,2,0} \rightarrow \mathbb{T}^{1,1,0}$ . In conclusion one obtains that

$$\text{Vec}_{\mathbb{Q}}^{2m}(\mathbb{T}^{1,2,0}) \simeq \mathbb{Z}_2 \oplus (2\mathbb{Z})^2, \quad \forall m \in \mathbb{N}.$$

This implies that one can build a  $\mathbb{Q}$ -structure on  $\mathbb{T}^{1,2,0}$  if and only if the underlying complex vector bundle has a first Chern class of type  $(2n, 2n', 0) \in \mathbb{Z}^3$ . Moreover, if this is the case, it is possible to build to inequivalent  $\mathbb{Q}$ -structures.

### - Cases with free involution -

As soon as  $c \neq 0$  the involution over  $\mathbb{T}^{a,b,c}$  turns out to be free. In this case the classification of “Quaternionic” vector bundles is simplified by two general results. The first of them concerns the possibility to reduce situations with  $c \geq 2$  to the *standard* situation  $c = 1$ .

**Proposition 4.9.** *For each  $a, b \in \mathbb{N} \cup \{0\}$  and  $c \geq 2$  there is an identification of  $\mathbb{Z}_2$ -spaces*

$$\mathbb{T}^{a,b,c} \simeq \mathbb{T}^{a+c-1,b,1}.$$

*Proof.* Consider the identifications  $\mathbb{S}^1 \equiv \mathbb{U}(1)$  and  $\mathbb{T}^d \equiv \mathbb{U}(1) \times \dots \times \mathbb{U}(1)$ . By using the group structure of  $\mathbb{U}(1)$  we can define maps  $f_c : \mathbb{T}^d \rightarrow \mathbb{T}^d$  of the type

$$f_c : (z_1, \dots, z_{d-c}, \dots, z_{d-1}, z_d) \mapsto (z_1, \dots, z_{d-c}z_d, \dots, z_{d-1}z_d, z_d).$$

Topologically these maps are homeomorphisms. Since  $\mathbb{S}^{1,0}$  is identifiable with  $\mathbb{U}(1)$  endowed with trivial involution  $z \mapsto z$  and  $\mathbb{S}^{0,1}$  is identifiable with  $\mathbb{U}(1)$  with antipodal involution  $z \mapsto -z$  one concludes that  $f_c$  is a  $\mathbb{Z}_2$ -homeomorphism between  $\mathbb{T}^{a,b,c}$  and  $\mathbb{T}^{a+c-1,b,1}$ . ■

The second result concerns involutive spaces with the product structure  $X \times \mathbb{S}^{0,2}$ .

**Proposition 4.10.** *For each involutive space  $(X, \tau)$  the FKMM-invariant*

$$\kappa : \text{Vec}_{\mathbb{Q}}^2(X \times \mathbb{S}^{0,2}) \longrightarrow H_{\mathbb{Z}_2}^2(X \times \mathbb{S}^{0,2}, \mathbb{Z}(1))$$

*is surjective. Moreover there is a bijection*

$$\text{Pic}_{\mathbb{Q}}(X \times \mathbb{S}^{0,2}) \simeq \text{Pic}_{\mathbb{R}}(X \times \mathbb{S}^{0,2}) \neq \emptyset$$

*which preserves the first Chern class.*

*Proof.* Since the involution on  $X \times \mathbb{S}^{0,2}$  is free the FKMM-invariant is given by the composition

$$\text{Vec}_{\mathbb{Q}}^2(X \times \mathbb{S}^{0,2}) \xrightarrow{\det} \text{Pic}_{\mathbb{R}}(X \times \mathbb{S}^{0,2}) \xrightarrow{c_1^{\mathbb{R}}} H_{\mathbb{Z}_2}^2(X \times \mathbb{S}^{0,2}, \mathbb{Z}(1))$$

(cf. Corollary 2.12 (2)). In order to show that  $\text{Pic}_{\mathbb{Q}}(X \times \mathbb{S}^{0,2}) \neq \emptyset$  let us introduce a special  $\mathbb{Q}$ -line bundle by mimicking the construction in Example 3.7. Consider the product line bundle  $X \times \mathbb{S}^{0,2} \times \mathbb{C} \rightarrow X \times \mathbb{S}^{0,2}$  endowed with the “Quaternionic” structure

$$\Theta_q : (x, k, \lambda) \mapsto (\tau(x), -k, q(x, k)\bar{\lambda}) \quad (x, k, \lambda) \in X \times \mathbb{S}^{0,2} \times \mathbb{C}$$

where the map  $q : X \times \mathbb{S}^{0,2} \rightarrow \mathbb{U}(1)$  must fulfill the constraint  $q(\tau(x), -k) = -q(x, k)$ . A possible choice is to fix  $q(x, k) = q_0(k)$  where  $q_0$  is the map described by (3.3). Let  $\mathcal{L}_0 \in \text{Pic}_{\mathbb{Q}}(X \times \mathbb{S}^{0,2})$  be this  $\mathbb{Q}$ -line bundle. By construction  $c_1(\mathcal{L}_0) = 0$  for the Chern class of the underlying complex line bundle. Let  $\tilde{\mathcal{L}}_0^2 := \mathcal{L}_0 \otimes \mathcal{L}_0$ . This  $\mathbb{Q}$ -line bundle agrees with the trivial element in  $\text{Pic}_{\mathbb{R}}(X \times \mathbb{S}^{0,2})$ . In fact the product structure of  $\tilde{\mathcal{L}}_0^2$  is evident by construction. Moreover, the  $\mathbb{R}$ -structure on  $\tilde{\mathcal{L}}_0^2$  is induced by the map  $q_0^2$  which can be factorized as  $q_0(k)^2 = \phi(-k)\phi(k)$  with  $\phi := i q_0$ . This shows that  $q_0^2$  is isomorphic through  $\phi$  to the constant map 1. For each  $\mathcal{L}_{\mathbb{R}} \in \text{Pic}_{\mathbb{R}}(X \times \mathbb{S}^{0,2})$  consider the rank two “Quaternionic” vector bundle over  $X \times \mathbb{S}^{0,2}$  given by

$$\mathcal{E}_{\mathbb{Q}} := (\mathcal{L}_{\mathbb{R}} \oplus (X \times \mathbb{S}^{0,2} \times \mathbb{C})) \otimes \tilde{\mathcal{L}}_0.$$

The associated determinant line bundle verifies

$$\det(\mathcal{E}_{\mathfrak{Q}}) \simeq \mathcal{L}_{\mathfrak{R}} \otimes (X \times \mathbb{S}^{0,2} \times \mathbb{C}) \otimes \tilde{\mathcal{L}}_0^2 \simeq \mathcal{L}_{\mathfrak{R}}$$

where we used the triviality of  $\tilde{\mathcal{L}}_0^2$ . This proves the surjectivity of  $\kappa$ . The last claim is a consequence of Corollary 3.2 and the existence of  $\tilde{\mathcal{L}}_0$ . ■

**Corollary 4.11.** *Let  $(X, \tau)$  be an involutive space that fulfills Assumption 2.1 and with dimension  $d \leq 2$ . Then the FKMM-invariant provides a bijection*

$$\text{Vec}_{\mathfrak{Q}}^m(X \times \mathbb{S}^{0,2}) \stackrel{\kappa}{\simeq} H_{\mathbb{Z}_2}^2(X \times \mathbb{S}^{0,2}, \mathbb{Z}(1)) \simeq \text{Pic}_{\mathfrak{R}}(X \times \mathbb{S}^{0,2})$$

valid for each  $m \in \mathbb{N}$  independently of its parity. In the odd rank case it is possible to normalize the FKMM-invariant  $\kappa$  in such a way that the bijection

$$\text{Vec}_{\mathfrak{Q}}^{2m-1}(X \times \mathbb{S}^{0,2}) \simeq \text{Pic}_{\mathfrak{R}}(X \times \mathbb{S}^{0,2})$$

preserves the Chern class of the underlying complex vector bundle.

*Proof.* The condition about the dimension of  $X$  assures the validity of the stable rank condition (cf. eq. (4.4) and eq. (4.9)). In the even rank case the bijectivity of  $\kappa$  is consequence of Theorem 4.7 (2) and Proposition 4.10. In the odd rank case the bijection follows from Theorem 4.8 and the existence of  $\tilde{\mathcal{L}}_0 \in \text{Pic}_{\mathfrak{Q}}(X \times \mathbb{S}^{0,2})$ . The use of  $\tilde{\mathcal{L}}_0$  as reference element for the construction of the bijection and the fact of  $c_1(\tilde{\mathcal{L}}_0) = 0$  imply the validity of the last claim. ■

• **The torus  $\mathbb{T}^{0,1,1}$ .** In view of Corollary 4.11 one has the bijection

$$\text{Vec}_{\mathfrak{Q}}^m(\mathbb{T}^{0,1,1}) \stackrel{\kappa}{\simeq} H_{\mathbb{Z}_2}^2(\mathbb{T}^{0,1,1}, \mathbb{Z}(1)) \simeq \mathbb{Z} \quad \forall m \in \mathbb{N}.$$

The computation of the cohomology group is provided in Proposition C.1. However, we can add more information by using the exact sequence [Go, Proposition 2.3]

$$\begin{array}{ccccccc} H_{\mathbb{Z}_2}^2(\mathbb{T}^{0,1,1}, \mathbb{Z}(1)) & \xrightarrow{f} & H^2(\mathbb{T}^2, \mathbb{Z}) & \longrightarrow & H_{\mathbb{Z}_2}^2(\mathbb{T}^{0,1,1}, \mathbb{Z}) & \longrightarrow & H_{\mathbb{Z}_2}^3(\mathbb{T}^{0,1,1}, \mathbb{Z}(1)) \\ \text{\tiny R} & & \text{\tiny R} & & \text{\tiny R} & & \text{\tiny R} \\ \mathbb{Z} & & \mathbb{Z} & & \mathbb{Z}_2 & & 0 \end{array}.$$

Here we used

$$\begin{aligned} H_{\mathbb{Z}_2}^2(\mathbb{T}^{0,1,1}, \mathbb{Z}) &\simeq H_{\mathbb{Z}_2}^2(\mathbb{S}^{0,2}, \mathbb{Z}) \oplus H_{\mathbb{Z}_2}^1(\mathbb{S}^{0,2}, \mathbb{Z}(1)) \simeq \mathbb{Z}_2 \\ H_{\mathbb{Z}_2}^3(\mathbb{T}^{0,1,1}, \mathbb{Z}(1)) &\simeq H_{\mathbb{Z}_2}^3(\mathbb{S}^{0,2}, \mathbb{Z}(1)) \oplus H_{\mathbb{Z}_2}^2(\mathbb{S}^{0,2}, \mathbb{Z}) \simeq 0 \end{aligned}$$

obtained by the application of the second isomorphism in (C.1). The exact sequence says that the forgetting map  $f$  acts as the multiplication by two and the composition with the first Chern class produces the identification

$$\text{Vec}_{\mathfrak{Q}}^m(\mathbb{T}^{0,1,1}) \stackrel{\kappa}{\simeq} H_{\mathbb{Z}_2}^2(\mathbb{T}^{0,1,1}, \mathbb{Z}(1)) \stackrel{c_1}{\simeq} 2\mathbb{Z} \quad \forall m \in \mathbb{N}.$$

We point out that the validity of this result in the odd rank case is a consequence of the existence of a bijection which preserves the first Chern classes as claimed by Corollary 4.11.

• **The tori  $\mathbb{T}^{1,0,1} \simeq \mathbb{T}^{0,0,2}$ .** Let us study the case  $\mathbb{T}^{1,0,1}$ . Corollary 4.11 and Proposition C.1 imply

$$\text{Vec}_{\mathfrak{Q}}^{2m}(\mathbb{T}^{1,0,1}) \stackrel{\kappa}{\simeq} H_{\mathbb{Z}_2}^2(\mathbb{T}^{1,0,1}, \mathbb{Z}(1)) \simeq \mathbb{Z}_2 \quad \forall m \in \mathbb{N}.$$

- **The torus**  $\mathbb{T}^{0,2,1}$ . From Corollary 4.11 and Proposition C.1 one obtains

$$\mathrm{Vec}_{\mathbb{Q}}^m(\mathbb{T}^{0,2,1}) \stackrel{\kappa}{\simeq} H_{\mathbb{Z}_2}^2(\mathbb{T}^{0,2,1}, \mathbb{Z}(1)) \simeq \mathbb{Z}^2 \quad \forall m \in \mathbb{N}.$$

A more precise description can be obtained by looking at the exact sequence [Go, Proposition 2.3]

$$\begin{array}{ccccccc} H_{\mathbb{Z}_2}^2(\mathbb{T}^{0,2,1}, \mathbb{Z}(1)) & \xrightarrow{f} & H^2(\mathbb{T}^3, \mathbb{Z}) & \longrightarrow & H_{\mathbb{Z}_2}^2(\mathbb{T}^{0,2,1}, \mathbb{Z}) & \longrightarrow & H_{\mathbb{Z}_2}^3(\mathbb{T}^{0,2,1}, \mathbb{Z}(1)) \\ \downarrow \scriptstyle \mathrm{R} & & \downarrow \scriptstyle \mathrm{R} & & \downarrow \scriptstyle \mathrm{R} & & \downarrow \scriptstyle \mathrm{R} \\ \mathbb{Z}^2 & & \mathbb{Z}^3 & & \mathbb{Z}_2^2 \oplus \mathbb{Z} & & \mathbb{Z}_2 \end{array}.$$

The cohomology groups can be computed by using the second isomorphism in (C.1) which gives

$$\begin{aligned} H_{\mathbb{Z}_2}^2(\mathbb{T}^{0,2,1}, \mathbb{Z}) &\simeq H_{\mathbb{Z}_2}^2(\mathbb{T}^{0,1,1}, \mathbb{Z}) \oplus H_{\mathbb{Z}_2}^1(\mathbb{S}^{0,2}, \mathbb{Z}(1)) \oplus H_{\mathbb{Z}_2}^0(\mathbb{S}^{0,2}, \mathbb{Z}) \simeq \mathbb{Z}_2^2 \oplus \mathbb{Z} \\ H_{\mathbb{Z}_2}^3(\mathbb{T}^{0,2,1}, \mathbb{Z}(1)) &\simeq H_{\mathbb{Z}_2}^3(\mathbb{T}^{0,1,1}, \mathbb{Z}(1)) \oplus H_{\mathbb{Z}_2}^2(\mathbb{T}^{0,1,1}, \mathbb{Z}) \simeq \mathbb{Z}_2. \end{aligned}$$

The exact sequence alone does not suffice for the determination of the forgetting map  $f$ . However, we can notice that  $H_{\mathbb{Z}_2}^2(\mathbb{T}^{0,2,1}, \mathbb{Z}(1)) \simeq \mathrm{Vec}_{\mathbb{Q}}^2(\mathbb{T}^{0,2,1})$  contains two  $2\mathbb{Z}$ -summands generated by the pullbacks of  $\mathrm{Vec}_{\mathbb{Q}}^2(\mathbb{T}^{0,1,1}) \simeq 2\mathbb{Z}$  under the two distinct projections  $\mathbb{T}^{0,2,1} \rightarrow \mathbb{T}^{0,1,1}$ . This fact implies that  $f$  must act as  $f : (n, n') \mapsto (2n, 2n', 0)$  with  $n, n' \in \mathbb{Z}$ . The composition with the first Chern class finally produces the identification

$$\mathrm{Vec}_{\mathbb{Q}}^m(\mathbb{T}^{0,2,1}) \stackrel{c_1}{\simeq} (2\mathbb{Z})^2 \quad \forall m \in \mathbb{N}.$$

- **The tori**  $\mathbb{T}^{1,1,1} \simeq \mathbb{T}^{0,1,2}$ . Let us discuss the case  $\mathbb{T}^{1,1,1}$ . Corollary 4.11 and Proposition C.1 imply

$$\mathrm{Vec}_{\mathbb{Q}}^m(\mathbb{T}^{1,1,1}) \stackrel{\kappa}{\simeq} H_{\mathbb{Z}_2}^2(\mathbb{T}^{1,1,1}, \mathbb{Z}(1)) \simeq \mathbb{Z}_2 \oplus \mathbb{Z}^2 \quad \forall m \in \mathbb{N}.$$

However, a deeper analysis shows that

$$\mathrm{Vec}_{\mathbb{Q}}^m(\mathbb{T}^{1,1,1}) \simeq \mathbb{Z}_2 \oplus (2\mathbb{Z})^2 \quad \forall m \in \mathbb{N}$$

where the  $\mathbb{Z}_2$ -summand is the result of the pullback of  $\mathrm{Vec}_{\mathbb{Q}}^m(\mathbb{T}^{1,0,1}) \simeq \mathbb{Z}_2$  under the projection  $\mathbb{T}^{1,1,1} \rightarrow \mathbb{T}^{1,0,1}$ . In the same way, the two  $2\mathbb{Z}$ -summands are generated by the pullbacks of  $\mathrm{Vec}_{\mathbb{Q}}^m(\mathbb{T}^{0,1,1}) \simeq 2\mathbb{Z}$  and  $\mathrm{Vec}_{\mathbb{Q}}^m(\mathbb{T}^{1,1,0}) \simeq 2\mathbb{Z}$  under the projections  $\mathbb{T}^{1,1,1} \rightarrow \mathbb{T}^{0,1,1}$  and  $\mathbb{T}^{1,1,1} \rightarrow \mathbb{T}^{1,1,0}$ , respectively.

- **The tori**  $\mathbb{T}^{2,0,1} \simeq \mathbb{T}^{1,0,2} \simeq \mathbb{T}^{0,0,3}$ . Let us focus on the case  $\mathbb{T}^{2,0,1}$ . From Corollary 4.11 and Proposition C.1 one obtains

$$\mathrm{Vec}_{\mathbb{Q}}^2(\mathbb{T}^{2,0,1}) \stackrel{\kappa}{\simeq} H_{\mathbb{Z}_2}^2(\mathbb{T}^{2,0,1}, \mathbb{Z}(1)) \simeq \mathbb{Z}_2^2 \quad m \in \mathbb{N}.$$

The two  $\mathbb{Z}_2$ -summands can be generated by the pullback of  $\mathrm{Vec}_{\mathbb{Q}}^m(\mathbb{T}^{1,0,1}) \simeq \mathbb{Z}_2$  under the two distinct projections  $\mathbb{T}^{2,0,1} \rightarrow \mathbb{T}^{1,0,1}$ .

## 5. APPLICATION TO PROTOTYPE MODELS OF TOPOLOGICAL QUANTUM SYSTEMS

Let  $\{\Sigma_0, \dots, \Sigma_4\} \in \mathrm{Mat}(\mathbb{C}^4)$  be an irreducible representation of the *Clifford algebra*  $Cl_{\mathbb{C}}(4)$  generated by the rules  $\Sigma_i \Sigma_j + \Sigma_j \Sigma_i = 2\delta_{i,j} \mathbb{1}_4$  for all  $i, j = 0, 1, \dots, 4$ . An explicit realization is given by

$$\begin{aligned} \Sigma_0 &:= \sigma_1 \otimes \sigma_3 & \Sigma_1 &:= \sigma_2 \otimes \sigma_3 & \Sigma_2 &:= \mathbb{1}_2 \otimes \sigma_1 \\ \Sigma_3 &:= \mathbb{1}_2 \otimes \sigma_2 & \Sigma_4 &:= \sigma_3 \otimes \sigma_3 \end{aligned}$$

where

$$\sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are the Pauli matrices. With this special choice the following relations hold true:

$$\Sigma_j^* = \Sigma_j, \quad \bar{\Sigma}_j := C \Sigma_j C = (-1)^j \Sigma_j, \quad \Sigma_0 \Sigma_1 \Sigma_2 \Sigma_3 \Sigma_4 = -\mathbb{1}_4.$$

where  $Cv = \bar{v}$  is the complex conjugation on  $\mathbb{C}^4$ . We need also the relations for the trace

$$\mathrm{Tr}_{\mathbb{C}^4}(\Sigma_j) = 0, \quad \mathrm{Tr}_{\mathbb{C}^4}(\Sigma_i \Sigma_j) = 4\delta_{i,j}. \quad (5.1)$$

Let  $J := \sigma_2 \otimes \mathbb{1}_2$  and  $\Theta := C \circ J = -J \circ C$ . Evidently  $J^* = J = J^{-1}$  and  $\Theta^2 = -\mathbb{1}_4$  which in turn implies  $\Theta^* = J \circ C$ . Then

$$\begin{cases} \Theta \Sigma_j \Theta^* = -\Sigma_j, & j = 0, 1, 3, 4. \\ \Theta \Sigma_2 \Theta^* = +\Sigma_2 \end{cases}$$

Given an involutive space  $(X, \tau)$  and five continuous functions  $F_j : X \rightarrow \mathbb{R}$ , where  $j = 0, 1, \dots, 4$ , one can build a *topological quantum system* (in the sense of [DG1, Definition 1.1])  $H : X \rightarrow \text{Mat}(\mathbb{C}^4)$  given by

$$H(x) := \sum_{j=0}^4 F_j(x) \Sigma_j. \quad (5.2)$$

By imposing the conditions

$$\begin{cases} F_j(\tau(x)) = -F_j(x), & j = 0, 1, 3, 4, \\ F_2(\tau(x)) = +F_2(x) \end{cases} \quad x \in X \quad (5.3)$$

one can verify that

$$\Theta H(x) \Theta^* = H(\tau(x)), \quad x \in X. \quad (5.4)$$

Equation (5.4) says that  $x \mapsto H(x)$  defines a topological quantum system with an *odd* time reversal symmetry in the sense of [DG1, Definition 1.2]. According to a commonly used notation, the class of these systems is denoted by the label *class AII* (see e. g. [AZ, SRFL]).

A simple computation provides

$$H(x)^2 = Q(x) \mathbb{1}_4, \quad Q(x) := \sum_{j=0}^4 F_j(x)^2$$

and, after assuming the *zero gap condition*  $Q > 0$ , one can define two projection-valued maps  $P_{\pm} : X \rightarrow \text{Mat}(\mathbb{C}^4)$  given by

$$P_{\pm}(x) := \frac{1}{2} \left( \mathbb{1}_4 \pm \sum_{j=0}^4 \frac{F_j(x)}{\sqrt{Q(x)}} \Sigma_j \right). \quad (5.5)$$

The projection relations  $P_{\pm}(x)^2 = P_{\pm}(x)$ ,  $P_{\pm}(x)P_{\mp}(x) = 0$  and  $P_{+}(x) \oplus P_{-}(x) = \mathbb{1}_4$  and the spectral relations  $[P_{\pm}(x), H(x)] = 0$  and  $P_{\pm}(x)H(x)P_{\pm}(x) = \pm \sqrt{Q(x)}P_{\pm}(x)$  can be directly checked. The conditions  $\text{Tr}_{\mathbb{C}^4} P_{\pm}(x) = 2$  shows that each map  $P_{\pm} : X \rightarrow \text{Mat}(\mathbb{C}^4)$  defines a continuous family of two dimensional planes in  $\mathbb{C}^4$ , namely a rank two complex vector bundle over  $X$ . Moreover, under the condition (5.3) which assures

$$\Theta P_{\pm}(x) \Theta^* = P_{\pm}(\tau(x)), \quad x \in X, \quad (5.6)$$

one obtains that the vector bundles  $\mathcal{E}_{\pm} \rightarrow X$  associated to  $P_{\pm}(x)$  are indeed “Quaternionic”. For more details about the construction of the (Bloch) vector bundles  $\mathcal{E}_{\pm}$  we refer to [DG1, Section 2 & Section 6]. Define  $f_j(x) := F_j(x)/\sqrt{Q(x)}$ . The family of these functions  $f_j : X \rightarrow \mathbb{R}$  is subjected to the pointwise constraint  $\sum_{j=0}^4 f_j(x)^2 = 1$ . Therefore, the vector valued map  $\phi := (f_0, f_1, \dots, f_4)$  provides a continuous map  $\phi : X \rightarrow \mathbb{S}^4$ . This map identifies uniquely the projections  $P_{\pm}(x)$  through the formula (5.5).

The last observation suggests the introduction of the *Hopf vector bundle*  $\mathcal{E}_{\text{Hopf}} \rightarrow \mathbb{S}^4$  defined by the projection-valued map  $P_{\text{Hopf}} : \mathbb{S}^4 \rightarrow \text{Mat}(\mathbb{C}^4)$  given by

$$P_{\text{Hopf}}(k_0, k_1, \dots, k_4) := \frac{1}{2} \left( \mathbb{1}_4 + \sum_{j=0}^4 k_j \Sigma_j \right). \quad (5.7)$$

The involution  $\Theta$  induces a “Quaternionic” structure on  $\mathcal{E}_{\text{Hopf}}$  provided that the base space  $\mathbb{S}^4$  is endowed with the involution  $(k_0, k_1, k_2, k_3, k_4) \mapsto (-k_0, -k_1, k_2, -k_3, -k_4)$ . This space, up to a reordering



of the indices, agrees with the *TR sphere*  $\mathbb{S}^{1,4}$  described in Section 1 (see also [DG1, Example 4.2]). Then,  $(\mathcal{E}_{\text{Hopf}}, \Theta)$  is an element of  $\text{Vec}_{\mathbb{Q}}^m(\mathbb{S}^{1,4})$ . This set has been classified in [DG2, Theorem 1.4 (i)] and we know that  $\text{Vec}_{\mathbb{Q}}^2(\mathbb{S}^{1,4}) \simeq \mathbb{Z}$  is completely classified by the second Chern class  $c_2$ . Moreover, the FKMM-invariant  $\kappa$  of elements in  $\text{Vec}_{\mathbb{Q}}^m(\mathbb{S}^{1,4})$  takes values in  $\mathbb{Z}_2$  and is fixed by the parity of  $c_2$ . The second Chern class of  $\mathcal{E}_{\text{Hopf}}$  is  $c_2(\mathcal{E}_{\text{Hopf}}) = 1$  (see e.g. [DG1, Remark 6.1]) and consequently one has

$$\kappa(\mathcal{E}_{\text{Hopf}}) = (-1)^{c_2(\mathcal{E}_{\text{Hopf}})} = -1.$$

Therefore, the Hopf vector bundle  $\mathcal{E}_{\text{Hopf}}$  provides an example of non-trivial  $\mathbb{Q}$ -bundle over  $\mathbb{S}^{1,4}$  with a non-trivial FKMM-invariant.

A comparison between (5.7) and (5.5) shows that the vector bundles  $\mathcal{E}_{\pm} \rightarrow X$  can be obtained from  $\mathcal{E}_{\text{Hopf}} \rightarrow \mathbb{S}^4$  by a pullback with respect a map  $\phi : X \rightarrow \mathbb{S}^4$  which amounts to the replacement of  $\pm f_j(x)$  with  $k_j$  in (5.7). This means that all possible vector bundle generated by a topological quantum system of type (5.2) can be labelled by continuous maps  $\phi : X \rightarrow \mathbb{S}^4$ . Moreover, since homotopy equivalent maps generated (via pullback) isomorphic vector bundles we obtain that the possible topological phases of the system (5.2) can be labelled by the set  $[X, \mathbb{S}^4]$  of classes of homotopy equivalence functions. However, if we want to respect the odd time reversal symmetry we need to take into account the constraints (5.3). These are compatible with the involution  $\theta_{1,4}$  since

$$\begin{aligned} \theta_{1,4}(f_0(x), f_1(x), f_2(x), f_3(x), f_4(x)) &= (-f_0(x), -f_1(x), f_2(x), -f_3(x), -f_4(x)) \\ &= (f_0(\tau(x)), f_1(\tau(x)), f_2(\tau(x)), f_3(\tau(x)), f_4(\tau(x))) \end{aligned}$$

and one deduces that the map  $\phi$  is indeed  $\mathbb{Z}_2$ -equivariant from  $(X, \tau)$  to  $\mathbb{S}^{1,4}$ . Therefore, the topology of a systems (5.2) with an odd time reversal symmetry imposed by (5.3) is encoded in the set  $[X, \mathbb{S}^{1,4}]_{\mathbb{Z}_2}$  of classes of  $\mathbb{Z}_2$ -homotopy equivalence of equivariant functions. In summary we showed that:

**Proposition 5.1** (Sufficient condition for non-trivial phases). *Each class AII topological quantum system given by a gapped Hamiltonian of type (5.2) subjected to the constraints (5.3) is obtained as the pullback of the Hopf  $\mathbb{Q}$ -bundle  $\mathcal{E}_{\text{Hopf}}$  with respect to a map  $[\phi] \in [X, \mathbb{S}^{1,4}]_{\mathbb{Z}_2}$ . In this sense  $[X, \mathbb{S}^{1,4}]_{\mathbb{Z}_2}$  provides a (generally non injective) complete set of labels for the topological phases of these systems. Finally, by naturality, the FKMM-invariant for these systems is obtained as*

$$\kappa(\phi^* \mathcal{E}_{\text{Hopf}}) = \phi^*(\kappa(\mathcal{E}_{\text{Hopf}})) = \phi^*(-1)$$

and is non-trivial whenever the equivariant map  $\phi : X \rightarrow \mathbb{S}^{1,4}$  induces a homomorphism

$$\phi^* : H_{\mathbb{Z}_2}^2(\mathbb{S}^{1,4} | (\mathbb{S}^{1,4})^\theta, \mathbb{Z}(1)) \longrightarrow H_{\mathbb{Z}_2}^2(X | X^\tau, \mathbb{Z}(1))$$

which is injective.

In Section B we computed

$$H_{\mathbb{Z}_2}^2(\mathbb{S}^{1,d} | (\mathbb{S}^{1,d})^\theta, \mathbb{Z}(1)) \simeq \begin{cases} 0 & \text{if } d = 0, 1 \\ \mathbb{Z}_2 & \text{if } d \geq 2. \end{cases} \quad (5.8)$$

Since the condition  $f_j \equiv 0$  is compatible with the constraints (5.3) we can try to extract information about this “simplified models”. First of all let us notice that if some of the  $f_j$ ’s vanishes then only a sub-vector bundle of the Hopf  $\mathbb{Q}$ -bundle  $\mathcal{E}_{\text{Hopf}}$  is responsible for the pullback. To be more precise, let us introduce the *embedding* maps  $\iota : \mathbb{S}^{1,d-1} \rightarrow \mathbb{S}^{1,d}$  which identify  $\mathbb{S}^{1,d-1}$  with the subset of  $\mathbb{S}^{1,d}$  fixed by the condition  $k_j = 0$  on one of the coordinates which behaves non-trivially under the involution  $\theta_{1,d}$  (according to the above notation this means  $j \neq 2$ ). Then, we have a sequence of equivariant maps

$$\mathbb{S}^{1,0} \xrightarrow{\iota} \mathbb{S}^{1,1} \xrightarrow{\iota} \dots \xrightarrow{\iota} \mathbb{S}^{1,d} \xrightarrow{\iota} \mathbb{S}^{1,d+1} \dots$$

and a related sequence of *restricted* Hopf  $\mathbb{Q}$ -bundle

$$\mathcal{E}_{\text{Hopf}}^{(iv-n)} := \iota^{n*} \mathcal{E}_{\text{Hopf}} \in \text{Vec}_{\mathbb{Q}}^2(\mathbb{S}^{1,4-n})$$

where we used the short notation  $\iota^n := \iota \circ \dots \circ \iota$  and the convention  $\mathcal{E}_{\text{Hopf}}^{(iv)} \equiv \mathcal{E}_{\text{Hopf}}$ . We know from the classification established in [DG2] that

$$\text{Vec}_{\mathbb{Q}}^2(\mathbb{S}^{1,0}) \simeq \text{Vec}_{\mathbb{Q}}^2(\mathbb{S}^{1,1}) = 0$$

which implies that  $\mathcal{E}_{\text{Hopf}}^{(i)}$  and  $\mathcal{E}_{\text{Hopf}}^{(0)}$  are both trivial. On the other hand we know that

$$\text{Vec}_{\mathbb{Q}}^2(\mathbb{S}^{1,3}) \simeq \text{Vec}_{\mathbb{Q}}^2(\mathbb{S}^{1,2}) = \mathbb{Z}_2$$

are completely classified by the FKMM-invariant and by naturality

$$\kappa(\mathcal{E}_{\text{Hopf}}^{(iii)}) = \iota^* \kappa(\mathcal{E}_{\text{Hopf}}) = \iota^*(-1), \quad \kappa(\mathcal{E}_{\text{Hopf}}^{(ii)}) = \iota^{2*} \kappa(\mathcal{E}_{\text{Hopf}}) = \iota^{2*}(-1).$$

Since the maps  $\iota : \mathbb{S}^{1,d-1} \rightarrow \mathbb{S}^{1,d}$  induce isomorphisms (see Lemma B.1)

$$\iota^* : H_{\mathbb{Z}_2}^2(\mathbb{S}^{1,d} | (\mathbb{S}^{1,d})^\theta, \mathbb{Z}(1)) \xrightarrow{\simeq} H_{\mathbb{Z}_2}^2(\mathbb{S}^{1,d-1} | (\mathbb{S}^{1,d-1})^\theta, \mathbb{Z}(1)), \quad d \geq 2$$

we can infer that

$$\mathcal{E}_{\text{Hopf}}^{(iii)} \in \text{Vec}_{\mathbb{Q}}^2(\mathbb{S}^{1,3}), \quad \mathcal{E}_{\text{Hopf}}^{(ii)} \in \text{Vec}_{\mathbb{Q}}^2(\mathbb{S}^{1,2})$$

are non-trivial representatives of the respective classes. These considerations have implications in the analysis of topological quantum systems of type (5.2).

**Proposition 5.2** (The case  $F_2 \neq 0$ ). *Consider class AII topological quantum systems given by gapped Hamiltonians of type (5.2) subjected to the constraints (5.3) and  $F_2 \neq 0$ . A necessary condition to produce a non-trivial topological phases is that at least two of the functions  $\{F_0, F_1, F_3, F_4\}$  have to be not identically zero. If this is the case, the set  $[X, \mathbb{S}^{1,4-n}]_{\mathbb{Z}_2}$  provides a (generally non injective) complete set of labels for the topological phases on these systems. Here  $n = 0, 1, 2$  represents the number of functions in  $\{F_0, F_1, F_3, F_4\}$  which are identically zero. Finally, by naturality, the FKMM-invariant of these systems is given by*

$$\kappa(\phi^* \mathcal{E}_{\text{Hopf}}^{(iv-n)}) = \phi^*(\kappa(\mathcal{E}_{\text{Hopf}}^{(iv-n)})) = \phi^*(-1), \quad n = 0, 1, 2$$

and is non-trivial whenever the equivariant map  $\phi : X \rightarrow \mathbb{S}^{1,4-n}$  induces a homomorphism

$$\phi^* : H_{\mathbb{Z}_2}^2(\mathbb{S}^{1,4-n} | (\mathbb{S}^{1,4-n})^\theta, \mathbb{Z}(1)) \longrightarrow H_{\mathbb{Z}_2}^2(X | X^\tau, \mathbb{Z}(1))$$

which is injective.

Let us now discuss the implications of the condition  $F_2 \equiv 0$ . In this case the Bloch vector bundles  $\mathcal{E}_\pm \rightarrow X$  are generated by the pullback with respect to a map  $\phi : X \rightarrow \mathbb{S}^{0,4} \subset \mathbb{S}^{1,4}$  where  $\mathbb{S}^{0,4}$  is the subspace fixed by the constraint  $k_2 = 0$ . In particular  $\mathbb{S}^{0,4}$  is the three dimensional sphere with antipodal free involution. The equivariant embedding  $\eta : \mathbb{S}^{0,4} \rightarrow \mathbb{S}^{1,4}$  produces by pullback the restricted Hopf  $\mathbb{Q}$ -bundle

$$\hat{\mathcal{E}}_{\text{Hopf}} := \eta^* \mathcal{E}_{\text{Hopf}} \in \text{Vec}_{\mathbb{Q}}^2(\mathbb{S}^{0,4}) = 0.$$

Since the Bloch vector bundles  $\mathcal{E}_\pm \rightarrow X$  subjected to the condition  $F_2 \equiv 0$  are obtained as pullbacks of  $\hat{\mathcal{E}}_{\text{Hopf}}$  which is trivial one obtains:

**Proposition 5.3** (The case  $F_2 \equiv 0$ ). *Consider class AII topological quantum systems given by gapped Hamiltonians of type (5.2) subjected to the constraints (5.3) and  $F_2 \equiv 0$ . These systems possesse only the trivial phase.*

## APPENDIX A. ABOUT THE EQUIVARIANT COHOMOLOGY OF THE CLASSIFYING SPACE

The cohomology of  $G_{2m}(\mathbb{C}^\infty)$  is the polynomial ring

$$H^\bullet(G_{2m}(\mathbb{C}^\infty), \mathbb{Z}) \simeq \mathbb{Z}[c_1, \dots, c_{2m}] \quad (\text{A.1})$$

generated by the *universal* Chern classes  $c_k \in H^{2k}(G_{2m}(\mathbb{C}^\infty), \mathbb{Z})$  [MS, Theorem 14.5]. The involution  $\rho$  on the space  $G_{2m}(\mathbb{C}^\infty)$  described in Section 2.6 induces a homomorphism  $\rho^*$  on each cohomology group  $H^k(G_{2m}(\mathbb{C}^\infty), \mathbb{Z})$ .  $\rho^*$  acts on the universal Chern classes as follows:

**Lemma A.1.** *It holds that*

$$\rho^*(c_k) = (-1)^k c_k, \quad k = 1, 2, \dots, 2m.$$

*Proof.* Since  $c_k$  is the  $k$ -th Chern class of the tautological vector bundle  $\mathcal{T}_{2m}^\infty \rightarrow G_{2m}(\mathbb{C}^\infty)$  it follows by naturality that  $\rho^*(c_k)$  is the  $k$ -th Chern class of the pullback vector bundle  $\rho^*(\mathcal{T}_{2m}^\infty)$ . The presence of the  $\mathbb{Q}$ -structure implies that the fibers of  $\mathcal{T}_{2m}^\infty$  and  $\rho^*(\mathcal{T}_{2m}^\infty)$  relative to a given point of the base space are related by an anti-linear transform. The result follows from the same argument as in [MS, Lemma 14.9]. ■

The equivariant cohomology of a fixed point  $\{*\}$  plays the role of the coefficient system for the equivariant cohomology of an equivariant space. We recall that (cf. [Go, Section 2.3] and [DG1, Section 5.1])

$$H_{\mathbb{Z}_2}^\bullet(\{*\}, \mathbb{Z}) = H^\bullet(\mathbb{R}P^\infty, \mathbb{Z}) \simeq \mathbb{Z}[t]/(2t) \quad (\text{A.2})$$

where the generator  $t \in H^2(\mathbb{R}P^\infty, \mathbb{Z}) \simeq \mathbb{Z}_2$  obeys  $2t = 0$  and

$$H_{\mathbb{Z}_2}^\bullet(\{*\}, \mathbb{Z}(1)) = H^\bullet(\mathbb{R}P^\infty, \mathbb{Z}(1)) \simeq t^{1/2} \mathbb{Z}[t]/(2t^{1/2}, 2t) \quad (\text{A.3})$$

where also  $t^{1/2} \in H^1(\mathbb{R}P^\infty, \mathbb{Z}(1)) \simeq \mathbb{Z}_2$  is subjected to  $2t^{1/2} = 0$ . The relation  $t^{1/2} \cup t^{1/2} = t$  given by the cup product provides the ring structure

$$H_{\mathbb{Z}_2}^\bullet(\{*\}, \mathbb{Z}) \oplus H_{\mathbb{Z}_2}^\bullet(\{*\}, \mathbb{Z}(1)) \simeq \mathbb{Z}[t^{1/2}]/(2t^{1/2}). \quad (\text{A.4})$$

The generators  $t^{1/2}$ ,  $t$  and  $t^{3/2} := t^{1/2} \cup t \in H_{\mathbb{Z}_2}^3(\{*\}, \mathbb{Z}(1))$  play an important role in the description of the (low dimensional) equivariant cohomology of the involutive space  $\hat{G}_{2m}(\mathbb{C}^\infty) \equiv (G_{2m}(\mathbb{C}^\infty), \rho)$ .

**Lemma A.2.** *The cohomology groups of the involutive space  $\hat{G}_{2m}(\mathbb{C}^\infty)$  up to the degree 4 are summarized in the following table:*

	$k = 0$	$k = 1$	$k = 2$	$k = 3$	$k = 4$
$H^k(G_{2m}(\mathbb{C}^\infty), \mathbb{Z})$	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}^2$
$H_{\mathbb{Z}_2}^k(\hat{G}_{2m}(\mathbb{C}^\infty), \mathbb{Z})$	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}^2 \oplus \mathbb{Z}_2$
$H_{\mathbb{Z}_2}^k(\hat{G}_{2m}(\mathbb{C}^\infty), \mathbb{Z}(1))$	0	$\mathbb{Z}_2$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$

Moreover, the generators of  $H_{\mathbb{Z}_2}^1(\hat{G}_{2m}(\mathbb{C}^\infty), \mathbb{Z}(1))$ ,  $H_{\mathbb{Z}_2}^2(\hat{G}_{2m}(\mathbb{C}^\infty), \mathbb{Z})$  and  $H_{\mathbb{Z}_2}^3(\hat{G}_{2m}(\mathbb{C}^\infty), \mathbb{Z}(1))$  can be identified with  $t^{1/2}$ ,  $t$  and  $t^{3/2}$ , respectively. Similarly, the generator of the  $\mathbb{Z}_2$ -summand in the group  $H_{\mathbb{Z}_2}^4(\hat{G}_{2m}(\mathbb{C}^\infty), \mathbb{Z})$  can be identified with  $t^2 := t \cup t$ .

*Proof.* The ordinary (non-equivariant) cohomology is described by (A.1). The computation of the equivariant cohomology requires the use of the Leray-Serre spectral sequence. For more details we refer to [DG3, Appendix D] and references therein. We know that there is a converging spectral sequence

$$E_2^{p,q} \Rightarrow H^{p+q}(\hat{G}_{2m}(\mathbb{C}^\infty), \mathbb{Z}(j))$$

which in page 2 is given by the *group cohomology* of  $\mathbb{Z}_2$ ,

$$E_2^{p,q} := H_{\text{group}}^p(\mathbb{Z}_2; H^q(G_{2m}(\mathbb{C}^\infty), \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}(j)),$$

where  $j = 0, 1$  and the coefficient systems  $H^q(G_{2m}(\mathbb{C}^\infty), \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}(j)$  are regarded as  $\mathbb{Z}_2$ -modules with respect to the involution  $\rho^*$  on  $H^q(G_{2m}(\mathbb{C}^\infty), \mathbb{Z})$  described in Lemma A.1 and the action  $n \mapsto (-1)^j n$  on  $\mathbb{Z}(j)$ . The page 2 is concentrated in the first quadrant, meaning that  $E_2^{p,q} = 0$  if  $p < 0$  or  $q < 0$ . In the case  $j = 1$  the values of the 2-page  $E_2^{p,q}$  are showed in the following table:

$q = 5$	0	0	0	0	0	0
$q = 4$	0	$\mathbb{Z}_2^2$	0	$\mathbb{Z}_2^2$	0	$\mathbb{Z}_2^2$
$q = 3$	0	0	0	0	0	0
$q = 2$	$\mathbb{Z}$	0	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	0
$q = 1$	0	0	0	0	0	0
$q = 0$	0	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$
$E_2^{p,q} (j = 1)$	$p = 0$	$p = 1$	$p = 2$	$p = 3$	$p = 4$	$p = 5$

Let us first notice that  $E_2^{p,2i+1} = 0$  as a consequence of the fact that the cohomology  $H^q(G_{2m}(\mathbb{C}^\infty), \mathbb{Z})$  is concentrated in even degree. This implies that the differentials  $d_2 : E_2^{p,q} \rightarrow E_2^{p+2,q-1}$  are trivial so that  $E_3^{p,q} \simeq E_2^{p,q}$ . Moreover, the differentials

$$d_3 : E_3^{p,2} \rightarrow E_3^{p+3,0}, \quad p = 0, 2, 4, \dots$$

have to be trivial since  $E_3^{p,0}$  must survive inside the direct summand  $E_2^{p,0} \simeq H_{\mathbb{Z}_2}^p(\{*\}, \mathbb{Z}(1))$  of the decomposition

$$H^p(\hat{G}_{2m}(\mathbb{C}^\infty), \mathbb{Z}(1)) \simeq H_{\mathbb{Z}_2}^p(\{*\}, \mathbb{Z}(1)) \oplus \tilde{H}^p(\hat{G}_{2m}(\mathbb{C}^\infty), \mathbb{Z}(1)) \quad (\text{A.5})$$

where  $\tilde{H}^\bullet$  denotes the relative cohomology with respect to a fixed point. In fact, if  $d_3$  were non trivial, some of the  $E_3^{p+3,0}$  would be killed producing a contradiction. For instance, suppose that  $d_3 : E_3^{2,2} \rightarrow E_3^{5,0}$  is non trivial. Then,  $E_4^{5,0} = 0$  and so  $E_4^{5,0} \simeq E_5^{5,0} \simeq \dots \simeq E_\infty^{5,0} = 0$ . However, from a general property of the spectral sequence  $E_\infty^{5,0}$  injects into  $H^5(\hat{G}_{2m}(\mathbb{C}^\infty), \mathbb{Z}(1))$ . Consider the following commutative diagram

$$\begin{array}{ccc} E_5^{p,0}(\{*\}) \simeq H_{\mathbb{Z}_2}^p(\{*\}, \mathbb{Z}(1)) \simeq \mathbb{Z}_2 & & \\ \swarrow j_1^* & & \searrow j_2^* \\ E_5^{5,0} & \xrightarrow{s} & H^5(\hat{G}_{2m}(\mathbb{C}^\infty), \mathbb{Z}(1)) \end{array} \quad (\text{A.6})$$

where  $s$  is the injection induced by the spectral sequence and  $j_2^*$  is the homomorphism induced by the collapsing (equivariant) map  $j : \hat{G}_{2m}(\mathbb{C}^\infty) \rightarrow \{*\}$ . Also  $j_2^*$  is injective due to the decomposition (A.5). The map  $j_1^*$  can be seen as the homomorphism induced by  $j$  between the spectral sequence associated to the equivariant cohomology of the fixed point  $E_2^{p,0}(\{*\}) \simeq E_\infty^{p,0}(\{*\}) \simeq H_{\mathbb{Z}_2}^\bullet(\{*\}, \mathbb{Z}(1))$  and the spectral sequence for the cohomology of  $\hat{G}_{2m}(\mathbb{C}^\infty)$ . Since  $s \circ j_1^* = j_2^*$  is injective also  $j_1^*$  has to be injective, proving that  $E_\infty^{5,0} \neq 0$ . This contradiction shows that  $d_3$  is trivial. The triviality of  $d_3$  implies that  $E_4^{p,q} \simeq E_2^{p,q}$  for all  $0 \leq p + q \leq 4$ . Moreover, all the differentials  $d_r : E_r^{p,q} \rightarrow E_r^{p+r,q-r+1}$  are automatically trivial for  $0 \leq p + q \leq 4$  and  $r \geq 5$ , so that one concludes  $E_4^{p,q} \simeq E_\infty^{p,q}$  in the range  $0 \leq p + q \leq 4$ . In this situation the extension problem

$$0 \longrightarrow F^{p+1} H^{p+q}(\hat{G}_{2m}(\mathbb{C}^\infty), \mathbb{Z}(1)) \longrightarrow F^p H^{p+q}(\hat{G}_{2m}(\mathbb{C}^\infty), \mathbb{Z}(1)) \longrightarrow E_\infty^{p,q} \longrightarrow 0$$

can be (easily) solved. Notice that, among the pairs  $0 \leq p + q \leq 4$  such that  $p + q = n$  there is at least one pair which gives a non-trivial  $E_4^{p,q}$ . In conclusion one finds that

$$\begin{aligned} H^0(\hat{G}_{2m}(\mathbb{C}^\infty), \mathbb{Z}(1)) &\simeq E_\infty^{0,0} \simeq 0 \\ H^1(\hat{G}_{2m}(\mathbb{C}^\infty), \mathbb{Z}(1)) &\simeq E_\infty^{1,0} \simeq \mathbb{Z}_2 \\ H^2(\hat{G}_{2m}(\mathbb{C}^\infty), \mathbb{Z}(1)) &\simeq E_\infty^{0,2} \simeq \mathbb{Z} \\ H^3(\hat{G}_{2m}(\mathbb{C}^\infty), \mathbb{Z}(1)) &\simeq E_\infty^{3,0} \simeq \mathbb{Z}_2 \\ H^4(\hat{G}_{2m}(\mathbb{C}^\infty), \mathbb{Z}(1)) &\simeq E_\infty^{2,2} \simeq \mathbb{Z}_2. \end{aligned}$$

The case  $j = 0$  can be proved in a similar way. In this case the values of the 2-page are summarized in the following table:

$q = 5$	0	0	0	0	0	0
$q = 4$	$\mathbb{Z}^2$	0	$\mathbb{Z}_2^2$	0	$\mathbb{Z}_2^2$	0
$q = 3$	0	0	0	0	0	0
$q = 2$	0	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$
$q = 1$	0	0	0	0	0	0
$q = 0$	$\mathbb{Z}$	0	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	0
$E_2^{p,q} (j = 0)$	$p = 0$	$p = 1$	$p = 2$	$p = 3$	$p = 4$	$p = 5$

As above we have that  $E_3^{p,q} \simeq E_2^{p,q}$  and one can show that the maps  $d_3 : E_3^{p,2} \rightarrow E_3^{p+3,0}$  are trivial for all  $p$  since  $E_2^{p,0} \simeq H_{\mathbb{Z}_2}^p(\{*\}, \mathbb{Z})$  has to survive to contribute as a direct summand of  $H_{\mathbb{Z}_2}^p(\hat{G}_{2m}(\mathbb{C}^\infty), \mathbb{Z})$  due to the presence of a fixed point. Therefore, one can determine  $E_\infty^{p,q}$  for all  $p + q \leq 3$ . For  $p + q = 4$  one does not have enough information to determine whether  $d_3 : E_3^{0,4} \rightarrow E_3^{3,2}$  is trivial or not. However, in any event, the kernel of this map is  $\mathbb{Z}^2$  and this is enough to compute  $E_\infty^{p,q}$  also for all  $p + q = 4$ . Since  $E_3^{4,0}$  must survive as the contribution from  $H_{\mathbb{Z}_2}^4(\{*\}, \mathbb{Z})$ , the extension problem for  $H_{\mathbb{Z}_2}^4(\hat{G}_{2m}(\mathbb{C}^\infty), \mathbb{Z})$  is trivial, and the computation is completed.

Finally we notice that the identification with the generators  $t^{1/2}$ ,  $t$ ,  $t^{3/2}$  and  $t^2$  is clear from the direct summand argument. ■

The last result has important consequences on the description of the cohomology of  $\hat{G}_{2m}(\mathbb{C}^\infty)$ .

**Proposition A.3.** *The group  $H_{\mathbb{Z}_2}^2(\hat{G}_{2m}(\mathbb{C}^\infty), \mathbb{Z}(1)) \simeq \mathbb{Z}$  is generated by the (universal) first “Real” Chern class*

$$c_1^{\Re} := c_1^{\Re}(\det(\mathcal{F}_{2m}^\infty))$$

*of the “Real” line bundle  $\det(\mathcal{F}_{2m}^\infty) \rightarrow \hat{G}_{2m}(\mathbb{C}^\infty)$  associated to the tautological  $\mathfrak{Q}$ -bundle. Moreover, under the map  $f : H_{\mathbb{Z}_2}^2(\hat{G}_{2m}(\mathbb{C}^\infty), \mathbb{Z}(1)) \rightarrow H^2(G_{2m}(\mathbb{C}^\infty), \mathbb{Z})$  which forgets the  $\mathbb{Z}_2$ -action the class  $c_1^{\Re}$  is mapped in the first universal Chern class  $c_1$ .*

*Proof.* The exact sequence [Go, Proposition 2.3]

$$H_{\mathbb{Z}_2}^1(\hat{G}_{2m}(\mathbb{C}^\infty), \mathbb{Z}) \xrightarrow{\delta} H_{\mathbb{Z}_2}^2(\hat{G}_{2m}(\mathbb{C}^\infty), \mathbb{Z}(1)) \xrightarrow{f} H^2(G_{2m}(\mathbb{C}^\infty), \mathbb{Z}) \xrightarrow{g} H_{\mathbb{Z}_2}^2(\hat{G}_{2m}(\mathbb{C}^\infty), \mathbb{Z}) \dots$$



and  $H_{\mathbb{Z}_2}^1(\hat{G}_{2m}(\mathbb{C}^\infty), \mathbb{Z}) = 0$  imply that  $f$  is injective. Moreover,

$$\xrightarrow{g} H_{\mathbb{Z}_2}^2(\hat{G}_{2m}(\mathbb{C}^\infty), \mathbb{Z}) \xrightarrow{\delta'} H_{\mathbb{Z}_2}^3(\hat{G}_{2m}(\mathbb{C}^\infty), \mathbb{Z}(1)) \longrightarrow H^3(G_{2m}(\mathbb{C}^\infty), \mathbb{Z}) \dots,$$

along with  $H^3(G_{2m}(\mathbb{C}^\infty), \mathbb{Z}) = 0$  and  $H_{\mathbb{Z}_2}^2(\hat{G}_{2m}(\mathbb{C}^\infty), \mathbb{Z}) \simeq \mathbb{Z}_2 \simeq H_{\mathbb{Z}_2}^3(\hat{G}_{2m}(\mathbb{C}^\infty), \mathbb{Z}(1))$ , implies that  $\delta'$  is an isomorphism. Therefore  $g = 0$  and

$$f : H_{\mathbb{Z}_2}^2(\hat{G}_{2m}(\mathbb{C}^\infty), \mathbb{Z}(1)) \xrightarrow{\simeq} H^2(G_{2m}(\mathbb{C}^\infty), \mathbb{Z})$$

is an isomorphism. The general behavior of the ‘‘Real’’ Chern classes under the forgetting map is

$$f : c_1^{\Re}(\det(\mathcal{T}_{2m}^\infty)) \mapsto c_1(\det(\mathcal{T}_{2m}^\infty)).$$

Finally, the equality  $c_1(\det(\mathcal{T}_{2m}^\infty)) = c_1(\mathcal{T}_{2m}^\infty)$  and the definition of universal Chern class  $c_1 = c_1(\mathcal{T}_{2m}^\infty)$  conclude the proof.  $\blacksquare$

Since the fixed point set  $\hat{G}_{2m}(\mathbb{C}^\infty)^\rho$  can be identified with  $G_m(\mathbb{H}^\infty)$  one has that

$$H^\bullet(\hat{G}_{2m}(\mathbb{C}^\infty)^\rho, \mathbb{Z}) \simeq H^\bullet(G_m(\mathbb{H}^\infty), \mathbb{Z}). \quad (\text{A.7})$$

The latter is given by the polynomial ring

$$H^\bullet(G_m(\mathbb{H}^\infty), \mathbb{Z}) \simeq \mathbb{Z}[q_1, \dots, q_m] \quad (\text{A.8})$$

generated by the *universal (symplectic) Pontryagin classes*  $q_k \in H^{4k}(G_m(\mathbb{H}^\infty), \mathbb{Z})$  [Hus, Chapter 17].

**Lemma A.4.** *It holds that*

$$H_{\mathbb{Z}_2}^\bullet(\hat{G}_{2m}(\mathbb{C}^\infty)^\rho, \mathbb{Z}) \oplus H_{\mathbb{Z}_2}^\bullet(\hat{G}_{2m}(\mathbb{C}^\infty)^\rho, \mathbb{Z}(1)) \simeq \mathbb{Z}[t^{1/2}, q_1, \dots, q_m]/(2t^{1/2})$$

where the class  $t^{1/2}$  agrees with the generator of  $H_{\mathbb{Z}_2}^1(\{*\}, \mathbb{Z}(1)) \simeq \mathbb{Z}_2$  and  $q_k \in H^{4k}(\hat{G}_{2m}(\mathbb{C}^\infty)^\rho, \mathbb{Z})$  are the symplectic Pontryagin classes (up to the identification (A.7)).

*Proof.* The involution on  $\hat{G}_{2m}(\mathbb{C}^\infty)^\rho \simeq G_m(\mathbb{H}^\infty)$  is trivial, hence

$$H_{\mathbb{Z}_2}^\bullet(\hat{G}_{2m}(\mathbb{C}^\infty)^\rho, \mathbb{Z}(j)) \simeq H^\bullet(G_m(\mathbb{H}^\infty) \times \mathbb{R}P^\infty, \mathbb{Z}(j)), \quad j = 0, 1$$

by definition. The Künneth formula and the absence of torsion in  $H^\bullet(G_m(\mathbb{H}^\infty))$  imply that

$$H_{\mathbb{Z}_2}^\bullet(\hat{G}_{2m}(\mathbb{C}^\infty)^\rho, \mathbb{Z}(j)) \simeq H_{\mathbb{Z}_2}^\bullet(\{*\}, \mathbb{Z}(j)) \otimes H^\bullet(G_m(\mathbb{H}^\infty), \mathbb{Z})$$

More precisely, a comparison with (A.2) and (A.3) provides that

$$H_{\mathbb{Z}_2}^\bullet(\hat{G}_{2m}(\mathbb{C}^\infty)^\rho, \mathbb{Z}(j)) \simeq \begin{cases} \mathbb{Z}[t, q_1, \dots, q_m]/(2t) & \text{if } j = 0 \\ t^{1/2} \mathbb{Z}[t, q_1, \dots, q_m]/(2t^{1/2}, 2t) & \text{if } j = 1 \end{cases}$$

where  $t^{1/2} \in H_{\mathbb{Z}_2}^1(\{*\}, \mathbb{Z}(1))$ ,  $t \in H^2(\{*\}, \mathbb{Z}) \simeq \mathbb{Z}_2$  and  $q_k \in H^{4k}(\hat{G}_{2m}(\mathbb{C}^\infty)^\rho, \mathbb{Z})$  are the symplectic Pontryagin classes. The final result is a consequence of (A.4).  $\blacksquare$

The low order cohomology of  $\hat{G}_{2m}(\mathbb{C}^\infty)^\rho$  is summarized in the following table:

	$k = 0$	$k = 1$	$k = 2$	$k = 3$	$k = 4$
$H^k(\hat{G}_{2m}(\mathbb{C}^\infty)^\rho, \mathbb{Z})$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
$H_{\mathbb{Z}_2}^k(\hat{G}_{2m}(\mathbb{C}^\infty)^\rho, \mathbb{Z})$	$\mathbb{Z}$	0	$\mathbb{Z}_2$	0	$\mathbb{Z} \oplus \mathbb{Z}_2$
$H_{\mathbb{Z}_2}^k(\hat{G}_{2m}(\mathbb{C}^\infty)^\rho, \mathbb{Z}(1))$	0	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	0

The relative equivariant cohomology of the pair  $\hat{G}_{2m}(\mathbb{C}^\infty)^\rho \hookrightarrow \hat{G}_{2m}(\mathbb{C}^\infty)$  can be computed with the help of the long exact sequence (2.6).

**Proposition A.5.** *It holds that*

	$k = 0$	$k = 1$	$k = 2$	$k = 3$	$k = 4$
$H_{\mathbb{Z}_2}^k(\hat{G}_{2m}(\mathbb{C}^\infty) \hat{G}_{2m}(\mathbb{C}^\infty)^\rho, \mathbb{Z}(1))$	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$

and the map

$$\delta_2 : H_{\mathbb{Z}_2}^2(\hat{G}_{2m}(\mathbb{C}^\infty)|\hat{G}_{2m}(\mathbb{C}^\infty)^\rho, \mathbb{Z}(1)) \xrightarrow{\simeq} H_{\mathbb{Z}_2}^2(\hat{G}_{2m}(\mathbb{C}^\infty), \mathbb{Z}(1))$$

is an isomorphism.

*Proof.* The restriction map  $\hat{G}_{2m}(\mathbb{C}^\infty)^\rho \hookrightarrow \hat{G}_{2m}(\mathbb{C}^\infty)$  induces homomorphisms

$$r : H_{\mathbb{Z}_2}^k(\hat{G}_{2m}(\mathbb{C}^\infty), \mathbb{Z}(1)) \longrightarrow H_{\mathbb{Z}_2}^k(\hat{G}_{2m}(\mathbb{C}^\infty)^\rho, \mathbb{Z}(1)).$$

The analysis in Lemma A.2 and Lemma A.4 shows that the homomorphisms in degree  $k = 1$  and  $k = 3$  are in fact the isomorphisms which fix the generators  $t^{1/2}$  and  $t^{3/2}$ , respectively. To conclude the proof one needs to inspect the long exact sequence (2.6) for the relative equivariant cohomology  $H_{\mathbb{Z}_2}^k(\hat{G}_{2m}(\mathbb{C}^\infty)|\hat{G}_{2m}(\mathbb{C}^\infty)^\rho, \mathbb{Z}(1))$ . The degree  $k = 0$  is trivial. In degree  $k = 1$  the exact sequence reads

$$0 \xrightarrow{\delta_1} H_{\mathbb{Z}_2}^1(\hat{G}_{2m}(\mathbb{C}^\infty)|\hat{G}_{2m}(\mathbb{C}^\infty)^\rho, \mathbb{Z}(1)) \xrightarrow{\delta_2} \mathbb{Z}_2 \xrightarrow[r \simeq]{} \mathbb{Z}_2 \xrightarrow{\delta_1}$$

and this shows that  $H_{\mathbb{Z}_2}^1(\hat{G}_{2m}(\mathbb{C}^\infty)|\hat{G}_{2m}(\mathbb{C}^\infty)^\rho, \mathbb{Z}(1)) = 0$ . In degree  $k = 2$  this implies

$$0 \xrightarrow{\delta_1} H_{\mathbb{Z}_2}^2(\hat{G}_{2m}(\mathbb{C}^\infty)|\hat{G}_{2m}(\mathbb{C}^\infty)^\rho, \mathbb{Z}(1)) \xrightarrow{\delta_2} H_{\mathbb{Z}_2}^2(\hat{G}_{2m}(\mathbb{C}^\infty), \mathbb{Z}(1)) \xrightarrow{r} 0$$

and in turn the isomorphism between  $H_{\mathbb{Z}_2}^2(\hat{G}_{2m}(\mathbb{C}^\infty)|\hat{G}_{2m}(\mathbb{C}^\infty)^\rho, \mathbb{Z}(1))$  and  $H_{\mathbb{Z}_2}^2(\hat{G}_{2m}(\mathbb{C}^\infty), \mathbb{Z}(1)) \simeq \mathbb{Z}$ . In degree  $k = 3$  one has

$$0 \xrightarrow{\delta_1} H_{\mathbb{Z}_2}^3(\hat{G}_{2m}(\mathbb{C}^\infty)|\hat{G}_{2m}(\mathbb{C}^\infty)^\rho, \mathbb{Z}(1)) \xrightarrow{\delta_2} \mathbb{Z}_2 \xrightarrow[r \simeq]{} \mathbb{Z}_2 \xrightarrow{\delta_1}$$

which implies  $H_{\mathbb{Z}_2}^3(\hat{G}_{2m}(\mathbb{C}^\infty)|\hat{G}_{2m}(\mathbb{C}^\infty)^\rho, \mathbb{Z}(1)) = 0$ . Finally, in degree  $k = 4$  the exact sequence reads

$$0 \xrightarrow{\delta_1} H_{\mathbb{Z}_2}^4(\hat{G}_{2m}(\mathbb{C}^\infty)|\hat{G}_{2m}(\mathbb{C}^\infty)^\rho, \mathbb{Z}(1)) \xrightarrow{\delta_2} \mathbb{Z}_2 \xrightarrow{r} 0$$

and so  $H_{\mathbb{Z}_2}^4(\hat{G}_{2m}(\mathbb{C}^\infty)|\hat{G}_{2m}(\mathbb{C}^\infty)^\rho, \mathbb{Z}(1)) \simeq \mathbb{Z}_2$ . ■

## APPENDIX B. ABOUT THE EQUIVARIANT COHOMOLOGY OF INVOLUTIVE SPHERES

**B.1. TR-involution.** Let  $\mathbb{S}^{1,d} := (\mathbb{S}^d, \theta_{1,d})$  be the  $d$ -dimensional sphere with *TR-involution*  $\theta_{1,d} : (k_0, k_1, \dots, k_d) \mapsto (k_0, -k_1, \dots, -k_d)$  (cf. [DG1, Example 4.2]). The equivariant cohomology groups  $H_{\mathbb{Z}_2}^k(\mathbb{S}^{1,d}, \mathbb{Z}(1))$  have been computed in [DG1, Section 5.4]:

$$H_{\mathbb{Z}_2}^{\text{even}}(\mathbb{S}^{1,d}, \mathbb{Z}(1)) \simeq 0 \quad H_{\mathbb{Z}_2}^{\text{odd}}(\mathbb{S}^{1,d}, \mathbb{Z}(1)) \simeq \begin{cases} \mathbb{Z}_2 \oplus \mathbb{Z} & \text{if } k = d \\ \mathbb{Z}_2 & \text{if } k < d \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{if } k > d. \end{cases} \quad (\text{B.1})$$

Moreover, since  $(\mathbb{S}^{1,d})^\theta = \{-1, +1\}_{\text{fixed}}$  consists of two fixed points (in each dimension), one has that

$$H_{\mathbb{Z}_2}^k((\mathbb{S}^{1,d})^\theta, \mathbb{Z}(1)) \simeq H_{\mathbb{Z}_2}^k(\{*\}, \mathbb{Z}(1)) \oplus H_{\mathbb{Z}_2}^k(\{*\}, \mathbb{Z}(1)) \simeq \begin{cases} \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{if } k \text{ is odd} \\ 0 & \text{if } k \text{ is even.} \end{cases}$$

The groups  $H_{\mathbb{Z}_2}^2(\mathbb{S}^{1,d}|\mathbb{S}^{1,d})^\theta, \mathbb{Z}(1))$  for  $d \geq 2$  have been computed in [DG2, Proposition A.1]:

$$H_{\mathbb{Z}_2}^2(\mathbb{S}^{1,d}|\mathbb{S}^{1,d})^\theta, \mathbb{Z}(1)) \simeq \mathbb{Z}_2, \quad d \geq 2.$$

When  $d = 0$  the equality  $\mathbb{S}^{1,0} = \{-1, +1\}_{\text{fixed}} = (\mathbb{S}^{1,d})^\theta$  immediately implies

$$H_{\mathbb{Z}_2}^2(\mathbb{S}^{1,0}|\mathbb{S}^{1,0})^\theta, \mathbb{Z}(1)) = 0.$$

The case  $d = 1$  can be studied with the exact sequence (2.6) which in this case reads

$$0 \longrightarrow H_{\mathbb{Z}_2}^1(\mathbb{S}^{1,1} | (\mathbb{S}^{1,1})^\theta, \mathbb{Z}(1)) \xrightarrow{\delta_2} \mathbb{Z}_2 \oplus \mathbb{Z} \xrightarrow{r} \mathbb{Z}_2 \oplus \mathbb{Z}_2 \xrightarrow{\delta_1} H_{\mathbb{Z}_2}^2(\mathbb{S}^{1,1} | (\mathbb{S}^{1,1})^\theta, \mathbb{Z}(1)) \longrightarrow 0.$$

The same argument as in the proof of [DG2, Proposition A.1] shows that  $r$  acts bijectively on the  $\mathbb{Z}_2$  summand. On the other hand each homomorphism  $\mathbb{Z} \rightarrow \mathbb{Z}_2$  has  $\mathbb{Z}$  as kernel. Thus

$$H_{\mathbb{Z}_2}^1(\mathbb{S}^{1,1} | (\mathbb{S}^{1,1})^\theta, \mathbb{Z}(1)) = \mathbb{Z}.$$

Finally, since the  $\mathbb{Z}_2$ -summand in  $\mathbb{Z}_2 \oplus \mathbb{Z}$  is diagonally embedded into  $\mathbb{Z}_2 \oplus \mathbb{Z}_2$  and the image of the  $\mathbb{Z}$ -summand is  $\{(1, -1), (-1, 1)\}$ , one concludes that  $r$  is surjective, and so

$$H_{\mathbb{Z}_2}^2(\mathbb{S}^{1,1} | (\mathbb{S}^{1,1})^\theta, \mathbb{Z}(1)) = 0.$$

Let  $\iota : \mathbb{S}^{1,d-1} \rightarrow \mathbb{S}^{1,d}$  be the *embedding* maps which identify  $\mathbb{S}^{1,d-1}$  with the subset of  $\mathbb{S}^{1,d}$  defined by the constraint  $k_d = 0$ .

**Lemma B.1.** *The homomorphisms*

$$\iota^* : H_{\mathbb{Z}_2}^2(\mathbb{S}^{1,d} | (\mathbb{S}^{1,d})^\theta, \mathbb{Z}(1)) \longrightarrow H_{\mathbb{Z}_2}^2(\mathbb{S}^{1,d-1} | (\mathbb{S}^{1,d-1})^\theta, \mathbb{Z}(1))$$

*induced by  $\iota : \mathbb{S}^{1,d-1} \rightarrow \mathbb{S}^{1,d}$  are isomorphisms for all  $d \geq 3$ .*

*Proof.* For  $d \geq 2$  the contribution to  $H_{\mathbb{Z}_2}^1(\mathbb{S}^{1,d}, \mathbb{Z}(1)) \simeq \mathbb{Z}_2$  is given by the summand  $H_{\mathbb{Z}_2}^1(\{*\}, \mathbb{Z}(1))$  where the fixed point  $* \in \mathbb{S}^{1,d}$  is preserved by the embedding. Therefore,  $\iota^* : H_{\mathbb{Z}_2}^1(\mathbb{S}^{1,d}, \mathbb{Z}(1)) \rightarrow H_{\mathbb{Z}_2}^1(\mathbb{S}^{1,d-1}, \mathbb{Z}(1))$  is an isomorphism. Also the maps  $\iota^* : H_{\mathbb{Z}_2}^1((\mathbb{S}^{1,d})^\theta, \mathbb{Z}(1)) \rightarrow H_{\mathbb{Z}_2}^1((\mathbb{S}^{1,d-1})^\theta, \mathbb{Z}(1))$  are isomorphisms since the fixed point set  $\{-1, +1\}_{\text{fixed}}$  is preserved by the embedding. The naturality of the exact sequence for pairs concludes the proof. ■

**B.2. Antipodal involution.** Let  $\mathbb{S}^{0,d+1} := (\mathbb{S}^d, \theta_{0,d+1})$  be the  $d$ -dimensional sphere with (free) *antipodal involution*  $\theta_{0,d+1} : (k_0, k_1, \dots, k_d) \mapsto (-k_0, -k_1, \dots, -k_d)$  (cf. [DG1, Example 4.1]). We want to compute the equivariant cohomology of  $\mathbb{S}^{0,d+1}$ . The ordinary integer cohomology of the sphere is given by

$$H^k(\mathbb{S}^d, \mathbb{Z}) \simeq \begin{cases} \mathbb{Z} & \text{if } k = 0, d \\ 0 & \text{if } k \neq 0, d. \end{cases}$$

The equivariant cohomology with integer coefficients can be computed by observing that  $\theta_{0,d+1}$  acts freely with orbit space  $\mathbb{S}^{0,d+1} / \theta_{0,d+1} \simeq \mathbb{R}P^d$ . therefore

$$H_{\mathbb{Z}_2}^k(\mathbb{S}^{0,d+1}, \mathbb{Z}) \simeq H^k(\mathbb{R}P^d, \mathbb{Z}) \simeq \begin{cases} \mathbb{Z} & \text{if } k = 0 \\ \mathbb{Z}_2 & \text{if } k \text{ even } 0 < k < d \\ \mathbb{Z} & \text{if } k = d \text{ odd} \\ \mathbb{Z}_2 & \text{if } k = d \text{ even} \\ 0 & \text{otherwise} \end{cases}$$

by construction. The equivariant cohomology with  $\mathbb{Z}(1)$  coefficients can be investigated by the help of the *Gysin exact sequence*.

**Proposition B.2.** *Let  $\mathbb{S}^{0,d+1}$  be the sphere endowed with the free antipodal involution. Then:*

(1) *In odd dimension,  $d = 2n + 1$ :*

$$H_{\mathbb{Z}_2}^k(\mathbb{S}^{0,2n+2}, \mathbb{Z}(1)) \simeq \begin{cases} \mathbb{Z}_2 & \text{if } k \text{ odd } 1 \leq k \leq 2n + 1 \\ 0 & \text{otherwise} . \end{cases}$$

(2) *In even dimension,  $d = 2n$ :*

$$H_{\mathbb{Z}_2}^k(\mathbb{S}^{0,2n+1}, \mathbb{Z}(1)) \simeq \begin{cases} \mathbb{Z}_2 & \text{if } k \text{ odd } 1 \leq k < 2n \\ \mathbb{Z} & \text{if } k = 2n \\ 0 & \text{otherwise} . \end{cases}$$

*Proof.* (1) Let  $\pi : \underline{\mathbb{R}}^{d+1} \rightarrow \{*\}$  be the  $\mathbb{Z}_2$ -equivariant real vector bundle with non-trivial  $\mathbb{Z}_2$ -action given by  $\theta_{0,d+1} : (k_0, \dots, k_d) \mapsto (-k_0, \dots, -k_d)$ . The  $d$ -dimensional antipodal sphere  $\mathbb{S}^{0,d+1}$  coincides with the sphere-bundle of  $\pi : \underline{\mathbb{R}}^{d+1} \rightarrow \{*\}$ , i. e.  $\mathbb{S}^{0,d+1} = \mathbb{S}(\underline{\mathbb{R}}^{d+1})$ . The use of the Gysin exact sequence requires the specification of the *equivariant orientability* and the *equivariant Euler class* of  $\pi : \underline{\mathbb{R}}^{d+1} \rightarrow \{*\}$ . If  $d$  is odd,  $\pi : \underline{\mathbb{R}}^{d+1} \rightarrow \{*\}$  is equivariant orientable since its first  $\mathbb{Z}_2$ -equivariant Stiefel-Whitney class vanishes, i. e.  $w_1^{\mathbb{Z}_2}(\underline{\mathbb{R}}^{d+1}) = 0$ . This can be proved by observing that

$$w_1^{\mathbb{Z}_2}(\underline{\mathbb{R}}^{d+1}) \in H_{\mathbb{Z}_2}^1(\{*\}, \mathbb{Z}_2) \simeq H^1(\mathbb{R}P^\infty, \mathbb{Z}_2) \simeq \text{Hom}_{\mathbb{Z}}(\mathbb{Z}_2, \mathbb{Z}_2) \simeq \mathbb{Z}_2$$

agrees with the homomorphism induced by  $\det(\theta_{0,d+1}) : \mathbb{Z}_2 \rightarrow \mathbb{Z}_2$  and  $\det(\theta_{0,d+1}) = +1$  when  $d$  is odd. According to the proof of [Go, Proposition 2.6], the  $\mathbb{Z}_2$ -equivariant Euler class  $\chi^{\mathbb{Z}_2}(\underline{\mathbb{R}}^1)$  of  $\pi : \underline{\mathbb{R}}^1 \rightarrow \{*\}$  can be identified with the class  $t^{1/2}$  which generates the ring (A.4), so that

$$\chi^{\mathbb{Z}_2}(\underline{\mathbb{R}}^{d+1}) = \chi^{\mathbb{Z}_2}(\underline{\mathbb{R}}^1)^{d+1} = t^{(d+1)/2} \in H_{\mathbb{Z}_2}^{d+1}(\{*\}, \mathbb{Z}) .$$

Now, we are ready to use the Gysin sequence for the  $\pi : \mathbb{S}^{0,d+1} = \mathbb{S}(\underline{\mathbb{R}}^{d+1}) \rightarrow \{*\}$ :

$$H_{\mathbb{Z}_2}^{k-d-1}(\{*\}, \mathbb{Z}(1)) \xrightarrow{\chi^\cup} H_{\mathbb{Z}_2}^k(\{*\}, \mathbb{Z}(1)) \xrightarrow{\pi^*} H_{\mathbb{Z}_2}^k(\mathbb{S}^{0,d+1}, \mathbb{Z}(1)) \xrightarrow{\pi_*} H_{\mathbb{Z}_2}^{k-d}(\{*\}, \mathbb{Z}(1)) \xrightarrow{\chi^\cup} \dots$$

where  $\chi^\cup$  denotes the cup product with  $\chi^{\mathbb{Z}_2}(\underline{\mathbb{R}}^{d+1})$ . The map  $\pi^*$  turns out to be an isomorphism when  $k \leq d$ . Moreover,  $\chi^\cup$  is an isomorphism for  $k \geq d+1$  and this shows that  $H_{\mathbb{Z}_2}^k(\mathbb{S}^{0,d+1}, \mathbb{Z}(1)) = 0$  for  $k \geq d+1$ .

(2) When  $d$  is even the main difference is that the  $\mathbb{Z}_2$ -vector bundle  $\pi : \underline{\mathbb{R}}^{d+1} \rightarrow \{*\}$  is *not* equivariantly orientable since  $w_1^{\mathbb{Z}_2}(\underline{\mathbb{R}}^{d+1}) \simeq \det(\theta_{0,d+1}) = -1$ . This implies that the relevant Gysin sequence reads

$$H_{\mathbb{Z}_2}^{k-d-1}(\{*\}, \mathbb{Z}) \xrightarrow{\chi^\cup} H_{\mathbb{Z}_2}^k(\{*\}, \mathbb{Z}(1)) \xrightarrow{\pi^*} H_{\mathbb{Z}_2}^k(\mathbb{S}^{0,d+1}, \mathbb{Z}(1)) \xrightarrow{\pi_*} H_{\mathbb{Z}_2}^{k-d}(\{*\}, \mathbb{Z}) \xrightarrow{\chi^\cup} \dots$$

and, as in the odd case, one deduces that  $\pi^*$  is an isomorphism for  $k \leq d$  and  $H_{\mathbb{Z}_2}^k(\mathbb{S}^{0,d+1}, \mathbb{Z}(1)) = 0$  for  $k \geq d+1$ . ■

Proposition B.2 produces the following table in low dimension:

$H_{\mathbb{Z}_2}^k(\mathbb{S}^{0,d+1}, \mathbb{Z}(1))$	$k = 0$	$k = 1$	$k = 2$	$k = 3$	$k = 4$	$k = 5$	$k = 6$	...
$d = 1$	0	$\mathbb{Z}_2$	0	0	0	0	0	...
$d = 2$	0	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	0	...
$d = 3$	0	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	0	0	0	...
$d = 4$	0	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	...
$d = 5$	0	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	0	...

**B.3. More general involutions.** Let  $\mathbb{S}^{p,q}$  be an involutive sphere of type  $(p, q)$  according to the notation introduced in Section 1. There is an identification of  $\mathbb{Z}_2$ -spaces

$$\mathbb{S}^{p,q} \simeq \underbrace{\mathbb{S}^{2,0} \wedge \dots \wedge \mathbb{S}^{2,0}}_{(p-1)\text{-times}} \wedge \underbrace{\mathbb{S}^{1,1} \wedge \dots \wedge \mathbb{S}^{1,1}}_{q\text{-times}}, \quad p \geq 1$$

where  $\wedge$  denotes the *smash product* of topological spaces [Hat, Chapter 0]. Moreover, the equivariant *suspension* isomorphism [Go, Section 2.4] provides

$$\tilde{H}_{\mathbb{Z}_2}^k(\mathbb{S}^{p,q}, \mathbb{Z}(j)) \simeq \tilde{H}_{\mathbb{Z}_2}^{k-q}(\mathbb{S}^{p,0}, \mathbb{Z}(j-q)) \simeq H_{\mathbb{Z}_2}^{k-q-(p-1)}(\{*\}, \mathbb{Z}(j-q))$$

where  $\tilde{H}_{\mathbb{Z}_2}^k$  is the *reduced* cohomology. Finally, let us recall that in presence of a fixed point  $\{*\} \in X^\tau$  the exact sequence (2.6) can be extended to the reduced cohomology theory and one obtains

$$\dots \longrightarrow H_{\mathbb{Z}_2}^k(X|X^\tau, \mathcal{Z}) \xrightarrow{\delta_2} \tilde{H}_{\mathbb{Z}_2}^k(X, \mathcal{Z}) \xrightarrow{r} \tilde{H}_{\mathbb{Z}_2}^k(X^\tau, \mathcal{Z}) \xrightarrow{\delta_1} H_{\mathbb{Z}_2}^{k+1}(X|X^\tau, \mathcal{Z}) \longrightarrow \dots \quad (\text{B.2})$$

With this information one can compute the equivariant cohomology of  $\mathbb{S}^{2,1}$ . The fixed point set  $(\mathbb{S}^{2,1})^\theta \simeq \mathbb{S}^1$  is a circle and the suspension isomorphism says

$$\begin{aligned} \tilde{H}_{\mathbb{Z}_2}^k(\mathbb{S}^{2,1}, \mathbb{Z}(1)) &\simeq H_{\mathbb{Z}_2}^{k-2}(\{*\}, \mathbb{Z}) \simeq H^{k-2}(\mathbb{R}P^\infty, \mathbb{Z}) \\ \tilde{H}_{\mathbb{Z}_2}^k((\mathbb{S}^{2,1})^\theta, \mathbb{Z}(1)) &\simeq H_{\mathbb{Z}_2}^{k-1}(\{*\}, \mathbb{Z}(1)) \simeq H^{k-2}(\mathbb{R}P^\infty, \mathbb{Z}(1)). \end{aligned}$$

The values of  $H_{\mathbb{Z}_2}^k(\{*\}, \mathbb{Z}(j))$  are listed in (A.2) and (A.3). From (B.2) one extracts the following exact sequence

$$\begin{array}{ccccc} 0 \longrightarrow & H_{\mathbb{Z}_2}^2(\mathbb{S}^{2,1}|(\mathbb{S}^{2,1})^\theta, \mathbb{Z}(1)) & \xrightarrow{\delta_2} & \tilde{H}_{\mathbb{Z}_2}^2(\mathbb{S}^{2,1}, \mathbb{Z}(1)) & \xrightarrow{r} & \tilde{H}_{\mathbb{Z}_2}^2((\mathbb{S}^{2,1})^\theta, \mathbb{Z}(1)) \\ & \downarrow \text{R} & & \downarrow \text{R} & & \downarrow \text{R} \\ & ? & & \mathbb{Z} & & \mathbb{Z}_2 \end{array} \quad (\text{B.3})$$

which can be used to prove the following result:

**Proposition B.3.** *The map  $r$  in (B.3) is surjective and consequently there is an isomorphism of groups*

$$H_{\mathbb{Z}_2}^2(\mathbb{S}^{2,1}|(\mathbb{S}^{2,1})^\theta, \mathbb{Z}(1)) \simeq 2\mathbb{Z}$$

which is induced by the composition

$$H_{\mathbb{Z}_2}^2(\mathbb{S}^{2,1}|(\mathbb{S}^{2,1})^\theta, \mathbb{Z}(1)) \xrightarrow{\delta_2} H_{\mathbb{Z}_2}^2(\mathbb{S}^{2,1}, \mathbb{Z}(1)) \xrightarrow{f} H^2(\mathbb{S}^2, \mathbb{Z}) \xrightarrow{c_1} \mathbb{Z}$$

with  $f$  the map that forgets the  $\mathbb{Z}_2$ -action and  $c_1$  the first Chern class.

*Proof.* To understand the homomorphism  $r$  in (B.3) one identifies the bases of these cohomology groups by “Real” line bundles according to the Kahn’s isomorphism (2.5). The sequence of group isomorphisms

$$\tilde{H}_{\mathbb{Z}_2}^2((\mathbb{S}^{2,1})^\theta, \mathbb{Z}(1)) \simeq H_{\mathbb{Z}_2}^2(\mathbb{S}^{2,0}, \mathbb{Z}(1)) \simeq \text{Pic}_{\mathbb{R}}(\mathbb{S}^1) \simeq \mathbb{Z}_2 \quad (\text{B.4})$$

tells that the non-trivial element of  $\tilde{H}_{\mathbb{Z}_2}^2((\mathbb{S}^{2,1})^\theta, \mathbb{Z}(1))$  can be identified with the *Möbius bundle* on  $\mathbb{S}^1$  or, equivalently, with the “Real” line bundle  $\mathcal{L}_M = \mathbb{S}^1 \times \mathbb{C}$  with “Real” structure  $(k, \lambda) \mapsto (k, \phi(k)\bar{\lambda})$  where  $\phi : \mathbb{S}^1 \rightarrow \mathbb{U}(1)$  is the map given by  $\phi(k) := k_1 + i k_2$ . The first isomorphism in (B.4) can be verified directly by looking at the construction of the reduced cohomology groups (see e. g. [DG1, eq. 5.10]). The second isomorphism is justified by the fact that  $(\mathbb{S}^{2,1})^\theta$  coincides with  $\mathbb{S}^1$  endowed with the trivial involution and by [DG1, Proposition 4.5]. In a similar way one has the group isomorphisms

$$\tilde{H}_{\mathbb{Z}_2}^2(\mathbb{S}^{2,1}, \mathbb{Z}(1)) \simeq H_{\mathbb{Z}_2}^2(\mathbb{S}^{2,1}, \mathbb{Z}(1)) \simeq \text{Pic}_{\mathbb{R}}(\mathbb{S}^{2,1}).$$

The isomorphism  $\text{Pic}_{\mathbb{R}}(\mathbb{S}^{2,1}) \simeq \mathbb{Z}$  can be realized explicitly. Consider the map  $f : H_{\mathbb{Z}_2}^2(\mathbb{S}^{2,1}, \mathbb{Z}(1)) \rightarrow H^2(\mathbb{S}^2, \mathbb{Z})$  which forgets the  $\mathbb{Z}_2$ -action. Since  $H_{\mathbb{Z}_2}^1(\mathbb{S}^{2,1}, \mathbb{Z}) = H_{\mathbb{Z}_2}^2(\mathbb{S}^{2,1}, \mathbb{Z}) = 0$  one concludes from [Go, Proposition 2.3] that  $f$  is a bijection. A non trivial representative for a generator of  $H^2(\mathbb{S}^2, \mathbb{Z}) \simeq \text{Pic}_{\mathbb{C}}(\mathbb{S}^2)$  is provided by the line bundle  $\mathcal{L}_1 \rightarrow \mathbb{S}^2$  described by the family of *Hopf projections*  $k \mapsto P_{\text{Hopf}}(k)$  in equation (3.5). We know that  $\mathcal{L}_1$  has Chern class  $c_1(\mathcal{L}_1) = 1$ . Moreover,  $\mathcal{L}_1$  can be endowed with a “Real” structure over  $\mathbb{S}^{2,1}$  induced by

$$C P_{\text{Hopf}}(k_0, k_1, k_2) C = P_{\text{Hopf}}(k_0, -k_1, k_2)$$

where  $C$  implements the complex conjugation on  $\mathbb{C}^2$  (cf. Section 5). Set the “Real” line bundle  $(\mathcal{L}_1, C)$  as a representative for a generator of  $H_{\mathbb{Z}_2}^2(\mathbb{S}^{2,1}, \mathbb{Z}(1))$ . The inclusion  $\iota : \mathbb{S}^{2,0} \rightarrow \mathbb{S}^{2,1}$ , realized by  $\iota : (k_0, k_2) \mapsto (k_0, 0, k_2)$ , can be used to define the restricted (pullback) line bundle  $\iota^* \mathcal{L}_1 \in H_{\mathbb{Z}_2}^2(\mathbb{S}^{2,0}, \mathbb{Z}(1))$ . The latter is equivalently described by the family of “restricted” Hopf projections

$$\mathbb{S}^1 \simeq \mathbb{R}/2\pi\mathbb{Z} \ni \omega \mapsto \tilde{P}_{\text{Hopf}}(\omega) := \frac{1}{2} \begin{pmatrix} 1 + \cos(\omega) & \sin(\omega) \\ \sin(\omega) & 1 - \cos(\omega) \end{pmatrix}$$



through the parameterization  $k_0 \equiv \sin(\omega)$  and  $k_2 \equiv \cos(\omega)$ . There is a continuous section of normalized eigenvectors

$$\mathbb{R}/2\pi\mathbb{Z} \ni \omega \longmapsto s(\omega) := e^{-i\frac{\omega}{2}} \begin{pmatrix} \cos\left(\frac{\omega}{2}\right) \\ \sin\left(\frac{\omega}{2}\right) \end{pmatrix}.$$

We notice that the pre-factor  $e^{-i\frac{\omega}{2}}$  is needed to assure the continuity of the section. One can use  $s$  to fix a global trivialization

$$\mathbb{S}^1 \times \mathbb{C} \ni (\omega, \lambda) \xrightarrow{\psi} \lambda s(\omega) \in \iota^* \mathcal{L}_1.$$

Since the map  $\psi$  is compatible with the “Real” structure of  $\mathcal{L}_M$  it follows that  $\iota^* \mathcal{L}_1 \simeq \mathcal{L}_M$  as “Real” line bundles. This fact immediately implies that the restriction map  $r$  in (B.3) is surjective. ■

The case  $\mathbb{S}^{2,2}$  can be studied along the same lines. Again the fixed point set  $(\mathbb{S}^{2,2})^\theta \simeq \mathbb{S}^1$  is a circle and the suspension isomorphism provides

$$\begin{aligned} \tilde{H}_{\mathbb{Z}_2}^k(\mathbb{S}^{2,2}, \mathbb{Z}(1)) &\simeq H_{\mathbb{Z}_2}^{k-3}(\{*\}, \mathbb{Z}(1)) \simeq H^{k-3}(\mathbb{R}P^\infty, \mathbb{Z}(1)) \\ \tilde{H}_{\mathbb{Z}_2}^k((\mathbb{S}^{2,2})^\theta, \mathbb{Z}(1)) &\simeq H_{\mathbb{Z}_2}^{k-1}(\{*\}, \mathbb{Z}(1)) \simeq H^{k-2}(\mathbb{R}P^\infty, \mathbb{Z}(1)). \end{aligned}$$

In this case the (B.2) immediately provides

$$H_{\mathbb{Z}_2}^2(\mathbb{S}^{2,2}|(\mathbb{S}^{2,2})^\theta, \mathbb{Z}(1)) = 0. \quad (\text{B.5})$$

The fixed point set of  $\mathbb{S}^{3,1}$  is a two-dimensional sphere  $(\mathbb{S}^{3,1})^\theta \simeq \mathbb{S}^2$ . In this case the suspension isomorphism provides

$$\begin{aligned} \tilde{H}_{\mathbb{Z}_2}^k(\mathbb{S}^{3,1}, \mathbb{Z}(1)) &\simeq H_{\mathbb{Z}_2}^{k-3}(\{*\}, \mathbb{Z}) \simeq H^{k-3}(\mathbb{R}P^\infty, \mathbb{Z}) \\ \tilde{H}_{\mathbb{Z}_2}^k((\mathbb{S}^{3,1})^\theta, \mathbb{Z}(1)) &\simeq H_{\mathbb{Z}_2}^{k-2}(\{*\}, \mathbb{Z}(1)) \simeq H^{k-2}(\mathbb{R}P^\infty, \mathbb{Z}(1)) \end{aligned}$$

and the exact sequence (B.2) implies

$$H_{\mathbb{Z}_2}^2(\mathbb{S}^{3,1}|(\mathbb{S}^{3,1})^\theta, \mathbb{Z}(1)) = 0. \quad (\text{B.6})$$

#### APPENDIX C. ABOUT THE EQUIVARIANT COHOMOLOGY OF INVOLUTIVE TORI

In this section we study the equivariant cohomology of involutive tori of type  $\mathbb{T}^{a,b,c}$  introduced in Section 1. Note that as soon as  $c \geq 1$  the involution on  $\mathbb{T}^{a,b,c}$  is free. An important computational tool is provided by the Gysin exact sequence [Go, Corollary 2.11] which establishes the isomorphisms of groups

$$\begin{aligned} H_{\mathbb{Z}_2}^k(X \times \mathbb{S}^{2,0}, \mathbb{Z}(j)) &\simeq H_{\mathbb{Z}_2}^k(X, \mathbb{Z}(j)) \oplus H_{\mathbb{Z}_2}^{k-1}(X, \mathbb{Z}(j)) \\ H_{\mathbb{Z}_2}^k(X \times \mathbb{S}^{1,1}, \mathbb{Z}(j)) &\simeq H_{\mathbb{Z}_2}^k(X, \mathbb{Z}(j)) \oplus H_{\mathbb{Z}_2}^{k-1}(X, \mathbb{Z}(j-1)). \end{aligned} \quad (\text{C.1})$$

**C.1. The free-involution cases.** When  $c > 0$  the space  $\mathbb{T}^{a,b,c}$  has a free involution. Due to Proposition 4.9 we can focus our attention on the case  $c = 1$ .

**Proposition C.1.** *For each  $a, b \in \mathbb{N} \cup \{0\}$  there is a group isomorphism*

$$H_{\mathbb{Z}_2}^2(\mathbb{T}^{a,b,1}, \mathbb{Z}(1)) \simeq \mathbb{Z}_2^a \oplus \mathbb{Z}^{(a+1)b}.$$

*Proof.* An iterated use of the Gysin exact sequences (C.1) provides

$$H_{\mathbb{Z}_2}^2(\mathbb{T}^{a,b,1}, \mathbb{Z}(1)) \simeq H_{\mathbb{Z}_2}^2(\mathbb{T}^{0,b,1}, \mathbb{Z}(1)) \oplus H_{\mathbb{Z}_2}^1(\mathbb{T}^{0,b,1}, \mathbb{Z}(1))^{\oplus a} \oplus H_{\mathbb{Z}_2}^0(\mathbb{T}^{0,b,1}, \mathbb{Z}(1))^{\oplus \binom{a}{2}}$$

where  $H_{\mathbb{Z}_2}^{-j}(X, \mathbb{Z}(1)) \equiv 0$  by definition. Similarly, one has

$$\begin{aligned} H_{\mathbb{Z}_2}^2(\mathbb{T}^{0,b,1}, \mathbb{Z}(1)) &\simeq H_{\mathbb{Z}_2}^2(\mathbb{S}^{0,2}, \mathbb{Z}(1)) \oplus H_{\mathbb{Z}_2}^1(\mathbb{S}^{0,2}, \mathbb{Z})^{\oplus b} \oplus H_{\mathbb{Z}_2}^0(\mathbb{S}^{0,2}, \mathbb{Z}(1))^{\oplus \binom{b}{2}} \\ H_{\mathbb{Z}_2}^1(\mathbb{T}^{0,b,1}, \mathbb{Z}(1)) &\simeq H_{\mathbb{Z}_2}^1(\mathbb{S}^{0,2}, \mathbb{Z}(1)) \oplus H_{\mathbb{Z}_2}^0(\mathbb{S}^{0,2}, \mathbb{Z})^{\oplus b} \\ H_{\mathbb{Z}_2}^0(\mathbb{T}^{0,b,1}, \mathbb{Z}(1)) &\simeq H_{\mathbb{Z}_2}^0(\mathbb{S}^{0,2}, \mathbb{Z}(1)) \end{aligned}$$

due to the equality  $\mathbb{T}^{0,0,1} = \mathbb{S}^{0,2}$ . The cohomology of  $\mathbb{S}^{0,2}$  has been computed in Section B.2. ■

In low dimension one has:

$H_{\mathbb{Z}_2}^2(\mathbb{T}^{a,b,1}, \mathbb{Z}(1))$	$b = 0$	$b = 1$	$b = 2$
$a = 0$	0	$\mathbb{Z}$	$\mathbb{Z}^2$
$a = 1$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \oplus \mathbb{Z}^2$	$\mathbb{Z}_2 \oplus \mathbb{Z}^4$
$a = 2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2 \oplus \mathbb{Z}^3$	$\mathbb{Z}_2^2 \oplus \mathbb{Z}^6$

**C.2. The cases with non-empty fixed point sets.** The condition  $c = 0$  implies that  $\mathbb{T}^{a,b,0}$  has fixed points. Let us start with the two-dimensional torus  $\mathbb{T}^{1,1,0}$  which has a fixed point set given by the disjoint union of two circles,  $(\mathbb{T}^{1,1,0})^\tau \simeq \mathbb{S}^{2,0} \sqcup \mathbb{S}^{2,0}$ . A repeated use of the Gysin exact sequences (C.1) provides

$$\begin{aligned} H_{\mathbb{Z}_2}^k(\mathbb{T}^{1,1,0}, \mathbb{Z}(j)) &\simeq H_{\mathbb{Z}_2}^k(\mathbb{S}^{2,0}, \mathbb{Z}(j)) \oplus H_{\mathbb{Z}_2}^{k-1}(\mathbb{S}^{2,0}, \mathbb{Z}(j-1)) \\ &\simeq H_{\mathbb{Z}_2}^k(\{*\}, \mathbb{Z}(j)) \oplus H_{\mathbb{Z}_2}^{k-1}(\{*\}, \mathbb{Z}(j)) \oplus H_{\mathbb{Z}_2}^{k-1}(\{*\}, \mathbb{Z}(j-1)) \oplus H_{\mathbb{Z}_2}^{k-2}(\{*\}, \mathbb{Z}(j-1)). \end{aligned}$$

The equivariant cohomology of the fixed point set is computed by

$$\begin{aligned} H_{\mathbb{Z}_2}^k((\mathbb{T}^{1,1,0})^\tau, \mathbb{Z}(j)) &\simeq H_{\mathbb{Z}_2}^k(\mathbb{S}^{2,0}, \mathbb{Z}(j)) \oplus H_{\mathbb{Z}_2}^k(\mathbb{S}^{2,0}, \mathbb{Z}(j)) \\ &\simeq H_{\mathbb{Z}_2}^k(\{*\}, \mathbb{Z}(j))^{\oplus 2} \oplus H_{\mathbb{Z}_2}^{k-1}(\{*\}, \mathbb{Z}(j))^{\oplus 2}. \end{aligned}$$

These computations allow to extract from (2.6) the exact sequence

$$\begin{array}{ccccc} \mathbb{Z}_2^2 & \longrightarrow & H_{\mathbb{Z}_2}^2(\mathbb{T}^{1,1,0} | (\mathbb{T}^{1,1,0})^\tau, \mathbb{Z}(1)) & \xrightarrow{\delta_2} & H_{\mathbb{Z}_2}^2(\mathbb{T}^{1,1,0}, \mathbb{Z}(1)) & \xrightarrow{r} & H_{\mathbb{Z}_2}^2((\mathbb{T}^{1,1,0})^\tau, \mathbb{Z}(1)) \\ & & \downarrow \text{R} & & \downarrow \text{R} & & \downarrow \text{R} \\ & & ? & & \mathbb{Z}_2 \oplus \mathbb{Z} & & \mathbb{Z}_2^2 \end{array} \quad (\text{C.2})$$

**Proposition C.2.** *The map  $r$  in (C.2) is surjective and consequently there is an isomorphism of groups*

$$H_{\mathbb{Z}_2}^2(\mathbb{T}^{1,1,0} | (\mathbb{T}^{1,1,0})^\tau, \mathbb{Z}(1)) \simeq 2\mathbb{Z} \quad (\text{C.3})$$

which is induced by the composition

$$H_{\mathbb{Z}_2}^2(\mathbb{T}^{1,1,0} | (\mathbb{T}^{1,1,0})^\tau, \mathbb{Z}(1)) \xrightarrow{\delta_2} H_{\mathbb{Z}_2}^2(\mathbb{T}^{1,1,0}, \mathbb{Z}(1)) \xrightarrow{f} H^2(\mathbb{T}^2, \mathbb{Z}) \xrightarrow{c_1} \mathbb{Z} \quad (\text{C.4})$$

where  $f$  is the map that forgets the  $\mathbb{Z}_2$ -action and  $c_1$  is the first Chern class.

*Proof.* The determination of  $H_{\mathbb{Z}_2}^2(\mathbb{T}^{1,1,0} | (\mathbb{T}^{1,1,0})^\tau, \mathbb{Z}(1))$  requires an accurate knowledge of the homomorphism  $r$  in the exact sequence (C.2). Let us start with the  $r$  in degree 1. In this case we can use the geometric identification (2.5) to write

$$H_{\mathbb{Z}_2}^1(\mathbb{T}^{1,1,0}, \mathbb{Z}(1)) \simeq [\mathbb{T}^{1,1,0}, \mathbb{S}^{1,1}]_{\mathbb{Z}_2} \xrightarrow{r} [(\mathbb{T}^{1,1,0})^\tau, \mathbb{S}^{1,1}]_{\mathbb{Z}_2} \simeq H_{\mathbb{Z}_2}^1((\mathbb{T}^{1,1,0})^\tau, \mathbb{Z}(1)).$$

A basis for

$$[(\mathbb{T}^{1,1,0})^\tau, \mathbb{S}^{1,1}]_{\mathbb{Z}_2} \simeq [\mathbb{S}^{2,0}, \mathbb{S}^{1,1}]_{\mathbb{Z}_2} \oplus [\mathbb{S}^{2,0}, \mathbb{S}^{1,1}]_{\mathbb{Z}_2} \simeq \mathbb{Z}_2^2$$

is provided by two copies of the constant map  $g : \mathbb{S}^{2,0} \rightarrow (-1, 0) \in \mathbb{S}^{1,1}$ . A basis for

$$[\mathbb{T}^{1,1,0}, \mathbb{S}^{1,1}]_{\mathbb{Z}_2} \simeq [\mathbb{S}^{1,1}, \mathbb{S}^{1,1}]_{\mathbb{Z}_2} \simeq \mathbb{Z}_2 \oplus \mathbb{Z} \quad (\text{C.5})$$

is given by the constant map  $\tilde{g} : \mathbb{S}^{2,0} \times \mathbb{S}^{1,1} \rightarrow (-1, 0) \in \mathbb{S}^{1,1}$  for the  $\mathbb{Z}_2$ -summand and by the projection  $\pi_2 : \mathbb{S}^{2,0} \times \mathbb{S}^{1,1} \rightarrow \mathbb{S}^{1,1}$  for the  $\mathbb{Z}$ -summand. Note that the first isomorphism in (C.5) is induced by  $H_{\mathbb{Z}_2}^1(\mathbb{T}^{1,1,0}, \mathbb{Z}(1)) \simeq H_{\mathbb{Z}_2}^1(\mathbb{S}^{1,1}, \mathbb{Z}(1))$  which follows from the use of the first Gysin exact sequence (C.1). This analysis shows that the map  $r$  is surjective in degree 1 and so the exact sequence (C.2) can be rewritten as

$$0 \longrightarrow H_{\mathbb{Z}_2}^2(\mathbb{T}^{1,1,0} | (\mathbb{T}^{1,1,0})^\tau, \mathbb{Z}(1)) \xrightarrow{\delta_2} \mathbb{Z}_2 \oplus \mathbb{Z} \xrightarrow{r} \mathbb{Z}_2^2. \quad (\text{C.6})$$

In particular  $\delta_2$  turns out to be an injection. The Kahn's isomorphism provides an easy way to describe the basis of

$$H_{\mathbb{Z}_2}^2(\mathbb{T}^{1,1,0}, \mathbb{Z}(1)) \simeq \text{Pic}_{\mathbb{R}}(\mathbb{S}^{2,0} \times \mathbb{S}^{1,1}) \simeq \mathbb{Z}_2 \oplus \mathbb{Z}.$$

The  $\mathbb{Z}_2$ -summand in  $\text{Pic}_{\mathbb{R}}(\mathbb{S}^{2,0} \times \mathbb{S}^{1,1})$  is given by the pullback under the projection  $\pi_1 : \mathbb{S}^{2,0} \times \mathbb{S}^{1,1} \rightarrow \mathbb{S}^{2,0}$  of the “Real” Möbius bundle  $\mathcal{L}_M \in \text{Pic}_{\mathbb{R}}(\mathbb{S}^{2,0})$  (cf. the proof of Proposition B.3). The construction of the generator of the  $\mathbb{Z}$ -summand requires a little of work. Let  $\mathbb{U}(1) \times \mathbb{R} \times \mathbb{C} \rightarrow \mathbb{U}(1) \times \mathbb{R}$  be the trivial complex line bundle over  $\mathbb{U}(1) \times \mathbb{R}$  endowed with a  $\mathbb{Z}$ -action  $\alpha_n : (z, x, \lambda) \mapsto (z, x + n, z^n \lambda)$  for all  $n \in \mathbb{Z}$ . Since the  $\mathbb{Z}$ -action is free on the base space one can build the *quotient* line bundle  $\mathcal{L}_{1,1} := (\mathbb{U}(1) \times \mathbb{R} \times \mathbb{C})/\alpha$  over  $\mathbb{U}(1) \times (\mathbb{R}/\mathbb{Z}) \simeq \mathbb{T}^2$ . This line bundle  $\mathcal{L}_{1,1} \rightarrow \mathbb{T}^2$  has Chern class  $c_1(\mathcal{L}_{1,1}) = 1$ . Moreover, it can be endowed with the “Real” structure  $\Theta : [(z, x, \lambda)] \mapsto [(z, -x, \bar{\lambda})]$  which converts  $\mathcal{L}_{1,1}$  into a non-trivial element of  $\text{Pic}_{\mathbb{R}}(\mathbb{S}^{2,0} \times \mathbb{S}^{1,1})$  with Chern class 1. This explicit construction proves, in particular, that the map  $f$  in (C.4) is surjective. A basis for

$$H_{\mathbb{Z}_2}^2((\mathbb{T}^{1,1,0})^\tau, \mathbb{Z}(1)) \simeq \text{Pic}_{\mathbb{R}}(\mathbb{S}^{2,0}) \oplus \text{Pic}_{\mathbb{R}}(\mathbb{S}^{2,0}) \simeq \mathbb{Z}_2^2$$

is given by two copies of the Möbius bundle  $\mathcal{L}_M$ . At this point it is enough to note that  $\mathcal{L}_{1,1}$  restricts to the trivial “Real” bundle on  $\mathbb{S}^{2,0} \times \{0\} \subset \mathbb{T}^{1,1,0}$  and to the Möbius bundle on  $\mathbb{S}^{2,0} \times \{\frac{1}{2}\} \subset \mathbb{T}^{1,1,0}$ . This fact means the map  $r$  in (C.6) is surjective, hence

$$H_{\mathbb{Z}_2}^2(\mathbb{T}^{1,1,0} | (\mathbb{T}^{1,1,0})^\tau, \mathbb{Z}(1)) \simeq \text{Ker} \left[ r : H_{\mathbb{Z}_2}^2(\mathbb{T}^{1,1,0}, \mathbb{Z}(1)) \rightarrow H_{\mathbb{Z}_2}^2((\mathbb{T}^{1,1,0})^\tau, \mathbb{Z}(1)) \right] \simeq 2\mathbb{Z}.$$

Finally, the injectivity of  $\delta_2$  and the surjectivity of  $f$  assure that the isomorphism (C.3) is realized by the composition of maps in (C.4).  $\blacksquare$

A similar result can be proved for the torus  $\mathbb{T}^{2,1,0}$ . Let us start again with the exact sequence

$$\begin{array}{ccccc} \mathbb{Z}_2^2 & \longrightarrow & H_{\mathbb{Z}_2}^2(\mathbb{T}^{2,1,0} | (\mathbb{T}^{2,1,0})^\tau, \mathbb{Z}(1)) & \xrightarrow{\delta_2} & H_{\mathbb{Z}_2}^2(\mathbb{T}^{2,1,0}, \mathbb{Z}(1)) & \xrightarrow{r} & H_{\mathbb{Z}_2}^2((\mathbb{T}^{2,1,0})^\tau, \mathbb{Z}(1)) \\ & & \downarrow \text{R} & & \downarrow \text{R} & & \downarrow \text{R} \\ & & ? & & \mathbb{Z}_2^2 \oplus \mathbb{Z}^2 & & \mathbb{Z}_2^4 \end{array} \quad (\text{C.7})$$

The computation of the cohomology groups follows by a repeated use of the Gysin exact sequences (C.1) along with the fact that the fixed point set  $(\mathbb{T}^{2,1,0})^\tau \simeq \mathbb{T}^{2,0,0} \sqcup \mathbb{T}^{2,0,0}$  is the disjoint union of two two-dimensional tori.

**Proposition C.3.** *The map  $r$  in (C.7) is surjective and consequently there is an isomorphism of groups*

$$H_{\mathbb{Z}_2}^2(\mathbb{T}^{2,1,0} | (\mathbb{T}^{2,1,0})^\tau, \mathbb{Z}(1)) \simeq (2\mathbb{Z})^2 \quad (\text{C.8})$$

which is induced by the composition

$$H_{\mathbb{Z}_2}^2(\mathbb{T}^{2,1,0} | (\mathbb{T}^{2,1,0})^\tau, \mathbb{Z}(1)) \xrightarrow{\delta_2} H_{\mathbb{Z}_2}^2(\mathbb{T}^{2,1,0}, \mathbb{Z}(1)) \xrightarrow{f} H^2(\mathbb{T}^3, \mathbb{Z}) \xrightarrow{c_1} \mathbb{Z}^3 \quad (\text{C.9})$$

where  $f$  is the map that forgets the  $\mathbb{Z}_2$ -action and  $c_1$  is the first Chern class.

*Proof.* The proof follows the same arguments as in Proposition C.2. One needs information about the homomorphism  $r$  in degree 1 and 2. In degree 1 the use of the geometric identification (2.5) provides

$$H_{\mathbb{Z}_2}^1(\mathbb{T}^{2,1,0}, \mathbb{Z}(1)) \simeq [\mathbb{T}^{2,1,0}, \mathbb{S}^{1,1}]_{\mathbb{Z}_2} \xrightarrow{r} [(\mathbb{T}^{2,1,0})^\tau, \mathbb{S}^{1,1}]_{\mathbb{Z}_2} \simeq H_{\mathbb{Z}_2}^1((\mathbb{T}^{2,1,0})^\tau, \mathbb{Z}(1)).$$

A basis for

$$[(\mathbb{T}^{2,1,0})^\tau, \mathbb{S}^{1,1}]_{\mathbb{Z}_2} \simeq [\mathbb{T}^{2,0,0}, \mathbb{S}^{1,1}]_{\mathbb{Z}_2} \oplus [\mathbb{T}^{2,0,0}, \mathbb{S}^{1,1}]_{\mathbb{Z}_2} \simeq \mathbb{Z}_2^2$$

is given by two copies of the constant map  $g : \mathbb{T}^{2,0,0} \rightarrow (-1, 0) \in \mathbb{S}^{1,1}$ . A basis for

$$[\mathbb{T}^{2,1,0}, \mathbb{S}^{1,1}]_{\mathbb{Z}_2} \simeq [\mathbb{S}^{1,1}, \mathbb{S}^{1,1}]_{\mathbb{Z}_2} \simeq \mathbb{Z}_2 \oplus \mathbb{Z} \quad (\text{C.10})$$

is given by the constant map  $\tilde{g} : \mathbb{T}^{2,0,0} \times \mathbb{S}^{1,1} \rightarrow -1 \in \mathbb{S}^{1,1}$  for the  $\mathbb{Z}_2$ -summand and by the projection  $\pi_3 : \mathbb{T}^{2,0,0} \times \mathbb{S}^{1,1} \rightarrow \mathbb{S}^{1,1}$  for the  $\mathbb{Z}$ -summand. Notice that the first isomorphism in (C.10) is consequence of  $H_{\mathbb{Z}_2}^1(\mathbb{T}^{2,1,0}, \mathbb{Z}(1)) \simeq H_{\mathbb{Z}_2}^1(\mathbb{S}^{1,1}, \mathbb{Z}(1))$  which follows from the use of the first Gysin exact sequence

(C.1). This analysis says that the map  $r$  is surjective in degree 1 and the exact sequence (C.7) assumes the short form

$$0 \longrightarrow H_{\mathbb{Z}_2}^2(\mathbb{T}^{2,1,0} | (\mathbb{T}^{2,1,0})^\tau, \mathbb{Z}(1)) \xrightarrow{\delta_2} \mathbb{Z}_2^2 \oplus \mathbb{Z}^2 \xrightarrow{r} \mathbb{Z}_2^4. \quad (\text{C.11})$$

In particular  $\delta_2$  is an injection. In degree 2 we can use the Kahn's isomorphism

$$H_{\mathbb{Z}_2}^2(\mathbb{T}^{2,1,0}, \mathbb{Z}(1)) \simeq \text{Pic}_{\mathbb{R}}(\mathbb{S}^{2,0} \times \mathbb{S}^{2,0} \times \mathbb{S}^{1,1}) \simeq \mathbb{Z}_2^2 \oplus \mathbb{Z}^2.$$

The two  $\mathbb{Z}_2$ -summands are given by the pullback under the first and second projection  $\pi_j : \mathbb{S}^{2,0} \times \mathbb{S}^{2,0} \times \mathbb{S}^{1,1} \rightarrow \mathbb{S}^{2,0}$  of the “Real” Möbius bundle  $\mathcal{L}_M \in \text{Pic}_{\mathbb{R}}(\mathbb{S}^{2,0})$  (cf. the proof of Proposition B.3). The two  $\mathbb{Z}$ -summands are given by the pullback under the two different projections  $\tilde{\pi}_j : \mathbb{S}^{2,0} \times \mathbb{S}^{2,0} \times \mathbb{S}^{1,1} \rightarrow \mathbb{S}^{2,0} \times \mathbb{S}^{1,1}$  of the line bundle  $\mathcal{L}_{1,1} \rightarrow \mathbb{S}^{2,0} \times \mathbb{S}^{1,1}$  constructed in the proof of Proposition C.2. This explicit construction also shows that the  $f$  in (C.9) restricts to a bijection from the free part  $\mathbb{Z}^2$  of  $H_{\mathbb{Z}_2}^2(\mathbb{T}^{2,1,0}, \mathbb{Z}(1))$  to a direct summand  $\mathbb{Z}^2$  in  $H^2(\mathbb{T}^3, \mathbb{Z}) \simeq \mathbb{Z}^3$ . A basis for

$$H_{\mathbb{Z}_2}^2((\mathbb{T}^{2,1,0})^\tau, \mathbb{Z}(1)) \simeq \text{Pic}_{\mathbb{R}}(\mathbb{T}^{2,0,0}) \oplus \text{Pic}_{\mathbb{R}}(\mathbb{T}^{2,0,0}) \simeq \mathbb{Z}_2^4$$

is given by two copies of  $\pi_j^* \mathcal{L}_M$  for each  $j = 1, 2$ . At this point the same argument in the proof of Proposition C.2 can be adapted to show that the map  $r$  in (C.7) is surjective, hence

$$H_{\mathbb{Z}_2}^2(\mathbb{T}^{2,1,0} | (\mathbb{T}^{2,1,0})^\tau, \mathbb{Z}(1)) \simeq \text{Ker}[r : H_{\mathbb{Z}_2}^2(\mathbb{T}^{2,1,0}, \mathbb{Z}(1)) \rightarrow H_{\mathbb{Z}_2}^2((\mathbb{T}^{2,1,0})^\tau, \mathbb{Z}(1))] \simeq (2\mathbb{Z})^2.$$

Finally, the injectivity of  $\delta_2$  and the properties of  $f$  assure that the isomorphism (C.3) is realized by the sequence of maps in (C.9).  $\blacksquare$

Finally, let us investigate the involutive torus  $\mathbb{T}^{1,2,0}$ . In this case the exact sequence reads

$$\begin{array}{ccccc} \mathbb{Z}_2^4 & \longrightarrow & H_{\mathbb{Z}_2}^2(\mathbb{T}^{1,2,0} | (\mathbb{T}^{1,2,0})^\tau, \mathbb{Z}(1)) & \xrightarrow{\delta_2} & H_{\mathbb{Z}_2}^2(\mathbb{T}^{1,2,0}, \mathbb{Z}(1)) & \xrightarrow{r} & H_{\mathbb{Z}_2}^2((\mathbb{T}^{1,2,0})^\tau, \mathbb{Z}(1)) \\ & & \text{R} & & \text{R} & & \text{R} \\ & & ? & & \mathbb{Z}_2 \oplus \mathbb{Z}^2 & & \mathbb{Z}_2^4 \end{array} \quad (\text{C.12})$$

The computation of the cohomology groups follows from a repeated use of the Gysin exact sequences (C.1) along with the fact that

$$(\mathbb{T}^{1,2,0})^\tau \simeq \mathbb{S}^{2,0} \sqcup \mathbb{S}^{2,0} \sqcup \mathbb{S}^{2,0} \sqcup \mathbb{S}^{2,0}.$$

**Proposition C.4.** *There is an isomorphism of groups*

$$H_{\mathbb{Z}_2}^2(\mathbb{T}^{1,2,0} | (\mathbb{T}^{1,2,0})^\tau, \mathbb{Z}(1)) \simeq \mathbb{Z}_2 \oplus (2\mathbb{Z})^2 \quad (\text{C.13})$$

which is given by the direct sum of

$$\begin{aligned} \text{Coker}[r : H_{\mathbb{Z}_2}^1(\mathbb{T}^{1,2,0}, \mathbb{Z}(1)) \rightarrow H_{\mathbb{Z}_2}^1((\mathbb{T}^{1,2,0})^\tau, \mathbb{Z}(1))] &\simeq \mathbb{Z}_2 \\ \text{Ker}[r : H_{\mathbb{Z}_2}^2(\mathbb{T}^{1,2,0}, \mathbb{Z}(1)) \rightarrow H_{\mathbb{Z}_2}^2((\mathbb{T}^{1,2,0})^\tau, \mathbb{Z}(1))] &\simeq (2\mathbb{Z})^2. \end{aligned}$$

Moreover, in the sequence of maps

$$H_{\mathbb{Z}_2}^2(\mathbb{T}^{1,2,0} | (\mathbb{T}^{1,2,0})^\tau, \mathbb{Z}(1)) \xrightarrow{\delta_2} H_{\mathbb{Z}_2}^2(\mathbb{T}^{1,2,0}, \mathbb{Z}(1)) \xrightarrow{f} H^2(\mathbb{T}^3, \mathbb{Z}) \stackrel{c_1}{\simeq} \mathbb{Z}^3 \quad (\text{C.14})$$

$\delta_2$  is an injection of the free part of  $H_{\mathbb{Z}_2}^2(\mathbb{T}^{1,2,0} | (\mathbb{T}^{1,2,0})^\tau, \mathbb{Z}(1))$  into the free part of  $H_{\mathbb{Z}_2}^2(\mathbb{T}^{1,2,0}, \mathbb{Z}(1))$  and  $f$  is a bijection between the free part of  $H_{\mathbb{Z}_2}^2(\mathbb{T}^{1,2,0}, \mathbb{Z}(1))$  and a direct  $\mathbb{Z}^2$ -summand in  $H^2(\mathbb{T}^3, \mathbb{Z})$ .

*Proof.* Let us study the homomorphism  $r$  in degree 1 and 2. In degree 1 we have the geometric identification

$$H_{\mathbb{Z}_2}^1(\mathbb{T}^{1,2,0}, \mathbb{Z}(1)) \simeq [\mathbb{T}^{1,2,0}, \mathbb{S}^{1,1}]_{\mathbb{Z}_2} \xrightarrow{r} [(\mathbb{T}^{1,2,0})^\tau, \mathbb{S}^{1,1}]_{\mathbb{Z}_2} \simeq H_{\mathbb{Z}_2}^1((\mathbb{T}^{1,2,0})^\tau, \mathbb{Z}(1)).$$

A basis for

$$[(\mathbb{T}^{1,2,0})^\tau, \mathbb{S}^{1,1}]_{\mathbb{Z}_2} \simeq [\mathbb{S}^{2,0}, \mathbb{S}^{1,1}]_{\mathbb{Z}_2}^{\oplus 4} \simeq \mathbb{Z}_2^4$$

is provided by four copies of the constant map  $g : \mathbb{S}^{2,0} \rightarrow (-1, 0) \in \mathbb{S}^{1,1}$ . A basis for

$$[\mathbb{T}^{1,2,0}, \mathbb{S}^{1,1}]_{\mathbb{Z}_2} \simeq H^1(\mathbb{S}^{1,1}, \mathbb{Z}(1)) \oplus H^0(\mathbb{S}^{1,1}, \mathbb{Z}) \simeq \mathbb{Z}_2 \oplus \mathbb{Z}^2 \quad (\text{C.15})$$

is given by the constant map  $\tilde{g} : \mathbb{T}^{1,2,0} \rightarrow (-1, 0) \in \mathbb{S}^{1,1}$  for the  $\mathbb{Z}_2$ -summand (which comes from  $H^1(\mathbb{S}^{1,1}, \mathbb{Z}(1))$ ) and by the two distinct projections  $\mathbb{S}^{2,0} \times \mathbb{S}^{1,1} \times \mathbb{S}^{1,1} \rightarrow \mathbb{S}^{1,1}$  for the two  $\mathbb{Z}$ -summands. This proves that

$$\text{Coker}[r : H_{\mathbb{Z}_2}^1(\mathbb{T}^{1,2,0}, \mathbb{Z}(1)) \rightarrow H_{\mathbb{Z}_2}^1((\mathbb{T}^{1,2,0})^\tau, \mathbb{Z}(1))] \simeq \mathbb{Z}_2$$

and the exact sequence (C.12) assumes the short expression

$$0 \rightarrow \mathbb{Z}_2 \rightarrow H_{\mathbb{Z}_2}^2(\mathbb{T}^{1,2,0}|(\mathbb{T}^{1,2,0})^\tau, \mathbb{Z}(1)) \xrightarrow{\delta_2} \mathbb{Z}_2 \oplus \mathbb{Z}^2 \xrightarrow{r} \mathbb{Z}_2^4. \quad (\text{C.16})$$

Before to describe the degree 2 let us note that the forgetting map  $f : H_{\mathbb{Z}_2}^1(\mathbb{T}^{1,2,0}, \mathbb{Z}(1)) \rightarrow H^1(\mathbb{T}^3, \mathbb{Z})$  induces a bijection from the free part  $\mathbb{Z}^2$  of  $H_{\mathbb{Z}_2}^1(\mathbb{T}^{1,2,0}, \mathbb{Z}(1))$  and a  $\mathbb{Z}^2$ -summand in  $H^1(\mathbb{T}^3, \mathbb{Z}) \simeq \mathbb{Z}^3$ . In fact this last  $\mathbb{Z}^2$ -summand turns out to be generated by the two projections  $\mathbb{S}^{2,0} \times \mathbb{S}^{1,1} \times \mathbb{S}^{1,1} \rightarrow \mathbb{S}^{1,1}$  upon forgetting the  $\mathbb{Z}_2$ -action. In degree 2 the Kahn's isomorphism provides

$$H_{\mathbb{Z}_2}^2(\mathbb{T}^{1,2,0}, \mathbb{Z}(1)) \simeq \text{Pic}_{\mathbb{R}}(\mathbb{S}^{2,0} \times \mathbb{S}^{1,1} \times \mathbb{S}^{1,1}) \simeq \mathbb{Z}_2 \oplus \mathbb{Z}^2.$$

The  $\mathbb{Z}_2$ -summand is generated by the pullback of the “Real” Möbius bundle  $\mathcal{L}_M \in \text{Pic}_{\mathbb{R}}(\mathbb{S}^{2,0})$  under the projection  $\mathbb{S}^{2,0} \times \mathbb{S}^{1,1} \times \mathbb{S}^{1,1} \rightarrow \mathbb{S}^{2,0}$ . In fact  $H_{\mathbb{Z}_2}^2(\mathbb{S}^{2,0}, \mathbb{Z}(1))$  is contained in  $H_{\mathbb{Z}_2}^2(\mathbb{T}^{1,2,0}, \mathbb{Z}(1))$  as a direct summand by the splitting of the Gysin sequence (C.1). The two  $\mathbb{Z}$ -summands are generated by the pullbacks of  $\mathcal{L}_{1,1} \in \text{Pic}_{\mathbb{R}}(\mathbb{T}^{1,1,0})$  under the two distinct projections  $\mathbb{T}^{1,2,0} \rightarrow \mathbb{T}^{1,1,0}$ . By an exact sequence argument one can show that the forgetting map  $f : H_{\mathbb{Z}_2}^2(\mathbb{T}^{1,2,0}, \mathbb{Z}(1)) \rightarrow H^2(\mathbb{T}^3, \mathbb{Z})$  induces an injection from the free part  $\mathbb{Z}^2$  of  $H_{\mathbb{Z}_2}^2(\mathbb{T}^{1,2,0}, \mathbb{Z}(1))$  into  $H^2(\mathbb{T}^3, \mathbb{Z}) \simeq \mathbb{Z}^3$ . Actually, this injection is a bijection from the free part of  $H_{\mathbb{Z}_2}^2(\mathbb{T}^{1,2,0}, \mathbb{Z}(1))$  to a direct  $\mathbb{Z}^2$ -summand in  $H^2(\mathbb{T}^3, \mathbb{Z})$ . In fact this  $\mathbb{Z}^2$ -summand can be generated by the pullbacks of  $\mathcal{L}_{1,1} \in \text{Pic}_{\mathbb{R}}(\mathbb{T}^{1,1,0})$  under the two distinct projections  $\mathbb{T}^{1,2,0} \rightarrow \mathbb{T}^{1,1,0}$  upon forgetting the  $\mathbb{Z}_2$ -action. A basis for

$$H_{\mathbb{Z}_2}^2((\mathbb{T}^{1,2,0})^\tau, \mathbb{Z}(1)) \simeq \text{Pic}_{\mathbb{R}}(\mathbb{S}^{2,0})^{\oplus 4} \simeq \mathbb{Z}_2^4$$

is given by four copies of the pullback of the line bundle  $\mathcal{L}_M$  under the four distinct projections  $(\mathbb{T}^{1,2,0})^\tau \rightarrow \mathbb{S}^{2,0}$ . This analysis shows that

$$\text{Ker}[r : H_{\mathbb{Z}_2}^2(\mathbb{T}^{1,2,0}, \mathbb{Z}(1)) \rightarrow H_{\mathbb{Z}_2}^2((\mathbb{T}^{1,2,0})^\tau, \mathbb{Z}(1))] \simeq (2\mathbb{Z})^2.$$

The conclusions above imply that the group  $H_{\mathbb{Z}_2}^2(\mathbb{T}^{1,2,0}|(\mathbb{T}^{1,2,0})^\tau, \mathbb{Z}(1))$  fits in the exact sequence

$$0 \rightarrow \mathbb{Z}_2 \rightarrow H_{\mathbb{Z}_2}^2(\mathbb{T}^{1,2,0}|(\mathbb{T}^{1,2,0})^\tau, \mathbb{Z}(1)) \rightarrow (2\mathbb{Z})^2 \rightarrow 0 \quad (\text{C.17})$$

which is splitting since  $(2\mathbb{Z})^2$  is a free group and this concludes the proof.  $\blacksquare$

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